Hence, the general value of z is

$$z = K\left(\frac{\gamma^2-9}{\gamma^2-1}\right)^{\frac{1}{4}} \int_{\gamma_0} \frac{(\gamma+1)(\gamma+3)^2}{(\gamma-3)^3} e^{-3\gamma} d\gamma,$$

the constants of integration being K and γ_0, or what is the same thing,

$$z = \left(\frac{\gamma^2-9}{\gamma^2-1}\right)^{\frac{1}{4}} \left\{C + D\int_{\infty} \frac{(\gamma+1)(\gamma+3)^2}{(\gamma-3)^3} e^{-3\gamma} d\gamma\right\},$$

the corresponding value of Q being

$$Q = \tfrac{1}{4}(\gamma^2-1)(\gamma^2-9)\frac{1}{z}\frac{dz}{d\gamma},$$

which contains the single arbitrary constant $\frac{D}{C}$; when this vanishes we have the foregoing particular solution $Q=\gamma$.

I recall that the expression of γ is

$$\gamma = \sqrt{(5+2\beta)}, = \sqrt{\left\{5+2\left(\alpha+\frac{1}{\alpha}\right)\right\}}, = \frac{1}{\sqrt{(\alpha)}}\sqrt{\{(2+\alpha)(1+2\alpha)\}},$$

where α is connected with k by the relation

$$k^2 = \frac{\alpha^3(2+\alpha)}{1+2\alpha}.$$

ON A MECHANICAL REPRESENTATION OF THE SECOND ELLIPTIC INTEGRAL.

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LET $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation to an elliptic cylinder *BADE* (fig. 22), then the equations to its circular sections through the origin are

$$y\sqrt{\left(\frac{1}{b^2} - \frac{1}{a^2}\right)} = \pm\frac{z}{a}.$$

Let *BCD* be a circular section of the cylinder; its equation is

$$(y-b)\sqrt{(a^2-b^2)} + bz = 0,$$

whence $$PN = \frac{b-y}{b}\sqrt{(a^2-b^2)},$$

now $CO = \sqrt{(a^2-b^2)}$; therefore $PM = \frac{y}{b}\sqrt{(a^2-b^2)}$.

Supposing the elliptic cylinder to roll along a plane, the circular section *BPD* will trace out a curve on the plane. For determining this curve, take in the plane as axis of ξ, the straight line on which the dotted ellipse *EMD* rolls, as origin the point at which *EB* when in contact with the plane, intersects the axis of ξ. And measure η perpendicular to ξ. Then it is clear that

$$\eta = PM = \frac{y}{b}\sqrt{(a^2-b^2)},$$

$$\xi = \text{arc}\,ME =$$

$$\int_y^b \sqrt{\left(1+\frac{a^4y^2}{b^4x^2}\right)}\,dy = \frac{b}{\sqrt{(a^2-b^2)}}\int_\eta^{\sqrt{(a^2-b^2)}} \sqrt{\left(1+\frac{a^4y^2}{b^4x^2}\right)}\,d\eta\,;$$

therefore $$\frac{d\xi}{d\eta} = \frac{-b}{\sqrt{(a^2-b^2)}}\sqrt{\left(1+\frac{a^4y^2}{b^4x^2}\right)}.$$

Whence $$x^2(a^2-b^2)\left[\left(\frac{d\xi}{d\eta}\right)^2+1\right] = a^4\,;$$

therefore $$\frac{x^2}{a^2} = \frac{a^2}{a^2-b^2}\cdot\left[\left(\frac{d\xi}{d\eta}\right)^2+1\right]^{-1},$$

also $$\frac{y^2}{b^2} = \frac{\eta^2}{a^2-b^2}\,;$$

therefore $$\frac{a^2}{\left(\frac{d\xi}{d\eta}\right)^2+1} + \eta^2 = a^2-b^2\,;$$

therefore $$\frac{d\xi}{d\eta} = \pm\sqrt{\left(\frac{b^2+\eta^2}{a^2-b^2-\eta^2}\right)},$$

$$\xi = \pm\int\sqrt{\left(\frac{b^2+\eta^2}{a^2-b^2-\eta^2}\right)}\,d\eta.$$

Put $$\eta = \sqrt{(a^2-b^2)}\cos\theta = ae\cos\theta.$$

Then taking the lower sign

$$\xi = a\int\sqrt{(1-e^2\sin^2\theta)}\,d\theta.$$

Now, if two circles of radius a be fastened together with their planes parallel, and inclined to the line joining their centres at an angle α, they clearly form the circular sections of an elliptic cylinder, of which the major axis is a, and the eccentricity $\cos\alpha$.

Therefore, if two such circles be rolled along a sheet of paper, the equation to their track is

$$\eta = a \cos\alpha \cos\theta,$$

$$\xi = a \int\sqrt{(1 - \cos^2\alpha \sin^2\theta)}\,.d\theta,$$

ξ is the second elliptic integral, and therefore this contrivance is a mechanical method of drawing a curve in which the abscissæ are simple elliptic integrals of the second kind.

If the circles be fixed together like an ordinary pair of parallel rulers (fig. 23), they may be so arranged as to give the value of any elliptic integral of this kind, where e is less than unity.

In what follows, I will, for brevity, call the line joining the centres of the two discs L (see fig. 24).

Let it be required to evaluate $a \int_0^\theta \sqrt{(1 - e^2 \sin^2\theta)}\, d\theta$ for some given value of θ.

First set the discs so that $\cos\alpha = e$.

It is clear from spherical trigonometry that θ is the angle BCP in fig. 22, that is to say, θ is the angle between the projection of L on the disc, and the radius of the disc which goes to the point of contact with the plane; see fig. 24. Suppose both discs graduated about their centres like a protractor, to degrees, the two projections of L being the two lines of $0°$ on the two discs respectively.

Draw on a sheet of paper two parallel lines distant L apart, and intersect them by a perpendicular to either, AB (fig. 24). Place the discs so that the two $0°$ are at A and B respectively, and roll them until the two points of contact come to θ on the angle-scale of each disc; let C, D be then the points of contact. Join CXD, intersecting one of our parallel lines (say A) in X. Then AX is the required elliptic integral.

It is clear that our two parallel lines will be the same for all values of (e); so that if they were drawn once for all, and were graduated, the result might be read off at once.

The complete integral $a \int_0^{\frac{1}{2}\pi} \sqrt{(1 - e^2 \sin^2\theta)}\, d\theta$ is half the distance from node to node of the sinuous curve.

When e is small the curve approximates to a straight line when e is nearly unity, the curve approximates to a succession of semicircles.

If the circular wheels were replaced by elliptic wheels, so arranged as to be sections of a circular cylinder, we should obtain the simple harmonic curve $\eta = a \sin\dfrac{\xi}{a}$.