THE
ELEMENTS
OF
EUCLID,
VIZ.

THE FIRST SIX BOOKS,

TOGETHER WITH THE

ELEVENTH AND TWELFTH.

The Errors, by which THEON, or others, have long ago
vitiated thefe Books, are corrected,

And fome of EUCLID's Demonftrations are reftored.

ALSO

THE BOOK OF

EUCLID'S DATA,

In like manner corrected.

BY

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To this SIXTH EDITION are alfo annexed,

ELEMENTS of PLAIN and SPHERICAL TRIGONOMETRY.

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M,DCC,LXXXI.

TO THE

K I N G,

THIS EDITION

OF THE

PRINCIPAL BOOKS

OF THE

ELEMENTS OF EUCLID,

AND OF THE

BOOK OF HIS DATA,

IS MOST HUMBLY DEDICATED,

BY

HIS MAJESTY's

MOST DUTIFUL, AND

MOST DEVOTED

SUBJECT AND SERVANT,

ROBERT SIMSON.

PREFACE.

THE Opinions of the Moderns concerning the Author of the Elements of Geometry, which go under Euclid's name, are very different and contrary to one another. Peter Ramus afcribes the Propofitions, as well as their Demonftrations, to Theon; others think the Propofitions to be Euclid's, but that the Demonftrations are Theon's; and others maintain that all the Propofitions and their Demonftrations are Euclid's own. John Buteo and Sir Henry Savile are the Authors of greateft Note who affert this laft, and the greater part of Geometers have ever fince been of this Opinion, as they thought it the moft probable. Sir Henry Savile, after the feveral Arguments he brings to prove it, makes this Conclufion (Page 13. Praelect) " That, excepting a very few " Interpolations, Explications, and Additions, Theon altered " nothing in Euclid." But, by often confidering and comparing together the Definitions and Demonftrations as they are in the Greek Editions we now have, I found that Theon, or whoever was the Editor of the prefent Greek Text, by adding fome things, fuppreffing others, and mixing his own with Euclid's Demonftrations, had changed more things to the worfe than is commonly fuppofed, and thofe not of fmall moment, efpecially in the Fifth and Eleventh Books of the Elements, which this Editor has greatly vitiated; for inftance, by fubftituting a fhorter, but infufficient Demonftration of the 18th Prop. of the 5th Book, in place of the legitimate one which Euclid had given; and by taking out of this Book, befides other things, the good Definition which Eudoxus or Euclid had given of Compound Ratio, and giving an abfurd one in place of it in the 5th Definition of the 6th Book, which neither Euclid, Archimedes, Appollonius, nor any Geometer before Theon's time, ever made ufe of, and of which there is not to be found the leaft appearance in any of their Writings; and, as this Definition did much embarafs Beginners, and is quite ufelefs, it is now thrown out of the Elements, and another, which, without doubt, Euclid had given, is put in its proper place among the Definitions of the

5th

5th Book, by which the Doctrine of Compound Ratios is rendered plain and eafy. Befides, among the Definitions of the 11th Book, there is this, which is the 10th, viz. " Equal " and fimilar folid Figures are thofe which are contained by " fimilar Planes of the fame Number and Magnitude." Now, this Propofition is a Theorem, not a Definition; becaufe the equality of Figures of any kind muft be demonftrated, and not affumed; and, therefore, though this were a true Propofition, it ought to have been demonftrated. But, indeed, this Propofition, which makes the 10th Definition of the 11th Book, is not true univerfally, except in the cafe in which each of the folid angles of the Figures is contained by no more than three plane Angles; for, in other Cafes, two folid Figures may be contained by fimilar Planes of the fame Number and Magnitude, and yet be unequal to one another; as fhall be made evident in the Notes fubjoined to thefe Elements. In like manner, in the Demonftration of the 26th Prop. of the 11th Book, it is taken for granted, that thofe folid Angles are equal to one another which are contained by plain Angles of the fame Number and Magnitude, placed in the fame Order; but neither is this univerfally true, except in the cafe in which the folid Angles are contained by no more than three plain Angles; nor of this Cafe is there any Demonftration in the Elements we now have, though it be quite neceffary there fhould be one. Now, upon the 10th Definition of this Book depend the 25th and 28th Propofitions of it; and, upon the 25th and 26th depend other eight, viz. the 27th, 31ft, 32d, 33d, 34th, 36th, 37th, and 40th of the fame Book; and the 12th of the 12th Book depends upon the eighth of the fame, and this 8th, and the Corollary of Propofition 17th, and Prop. 18th of the 12th Book, depend upon the 9th Definition of the 11th Book, which is not a right Definition; becaufe there may be Solids contained by the fame number of fimilar plane Figures, which are not fimilar to one another, in the true Senfe of Similarity received by all Geometers; and all thefe Propofitions have, for thefe Reafons, been infufficiently demonftrated fince Theon's time hitherto. Befides, there are feveral other things, which have nothing of Euclid's accuracy, and which plainly fhew, that his Elements have been much corrupted by unfkilful Geometers; and, though thefe are not fo grofs as the others now mentioned, they ought by no means to remain uncorrected.

Upon thefe Accounts it appeared neceffary, and I hope will prove acceptable to all Lovers of accurate Reafoning, and of
Ma-

Mathematical Learning, to remove such Blemishes, and reſtore the principal Books of the Elements to their original Accuracy, as far as I was able; eſpecially ſince theſe Elements are the Foundation of a Science by which the Inveſtigation and Diſcovery of uſeful Truths, at leaſt in Mathematical Learning, is promoted as far as the limited Powers of the Mind allow; and which likewiſe is of the greateſt Uſe in the Arts both of Peace and War, to many of which Geometry is abſolutely neceſſary. This I have endeavoured to do, by taking away the inaccurate and falſe Reaſonings which unſkilful Editors have put into the place of ſome of the genuine Demonſtrations of Euclid, who has ever been juſtly celebrated as the moſt accurate of Geometers, and by reſtoring to him thoſe Things which Theon or others have ſuppreſſed, and which have theſe many ages been buried in Oblivion.

In this Sixth Edition, Ptolemy's Propoſition concerning a Property of quadrilateral Figures in a Circle is added at the End of the ſixth Book. Alſo the Note on the 29th Prop. Book 1ſt, is altered, and made more explicit, and a more general Demonſtration is given, inſtead of that which was in the Note on the 10th Definition of Book 11th; beſides, the Tranſlation is much amended by the friendly Aſſiſtance of a learned Gentleman.

To which are alſo added, the elements of Plane and Spherical Trigonometry, which are commonly taught after the Elements of Euclid.

THE
ELEMENTS
OF
EUCLID.

BOOK I.

DEFINITIONS.

I.

A Point is that which hath no parts, or which hath no mag- nitude.

II.

A line is length without breadth.

III.

The extremities of a line are points.

IV.

A ftraight line is that which lies evenly between its extreme points.

V.

A fuperficies is that which hath only length and breadth.

VI.

The extremities of a fuperficies are lines.

VII.

A plane fuperficies is that in which any two points being taken, the ftraight line between them lies wholly in that fuperficies.

VIII.

" A plane angle is the inclination of two lines to one another
" in a plane, which meet together, but are not in the fame
" direction."

IX.

A plane rectilineal angle is the inclination of two ftraight lines to one another, which meet together, but are not in the fame ftraight line.

N. B.

A

D

B   C E
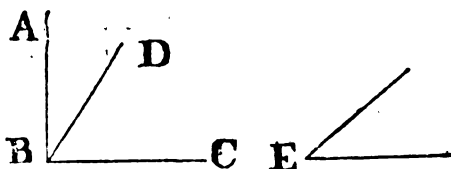
' N. B. When feveral angles are at one point B, any one
' of them is expreffed by three letters, of which the letter that
' is at the vertex of the angle, that is, at the point in which
' the ftraight lines that contain the angle meet one another, is
' put between the other two letters, and one of thefe two is
' fomewhere upon one of thofe ftraight lines, and the other
' upon the other line : Thus the angle which is contained by
' the ftraight lines AB, CB is named the angle ABC, or CBA ;
' that which is contained by AB, DB is named the angle
' ABD, or DBA ; and that which is contained by DB, CB is
' called the angle DBC, or CBD ; but, if there be only one angle
' at a point, it may be expreffed by a letter placed at that point ;
' as the Angle at E.'

X.

When a ftraight line ftanding on ano-
    ther ftraight line makes the adjacent
    angles equal to one another, each of
    the angles is called a right angle ;
    and the ftraight line which ftands
    on the other is called a perpendicular
    to it.

XI.
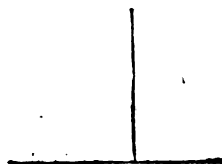
An obtufe angle is that which is greater than a right angle.

XII.

An acute angle is that which is lefs than a right angle.
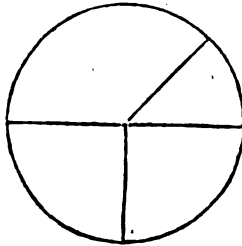
XIII.

" A term or boundary is the extremity of any thing."

XIV.

A figure is that which is inclofed by one or more boundaries.

XV.

XV.
A circle is a plane figure contained by one line, which is cal-
led the circumference, and is such that all straight lines
drawn from a certain point within the figure to the circum-
ference, are equal to one another :

XVI.
And this point is called the centre of the circle.

XVII.
A diameter of a circle is a straight line drawn through the
centre, and terminated both ways by the circumference.
See N.

XVIII.
A semicircle is the figure contained by a diameter and the part
of the circumference cut off by the diameter.

XIX.
" A segment of a circle is the figure contained by a straight
" line, and the circumference it cuts off."

XX.
Rectilineal figures are those which are contained by straight
lines.

XXI.
Trilateral figures, or triangles, by three straight lines.

XXII.
Quadrilateral, by four straight lines.

XXIII.
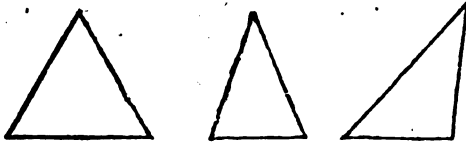Multilateral figures, or polygons, by more than four straight
lines.

XXIV.
Of three sided figures, an equilateral triangle is that which has
three equal sides.

XXV.
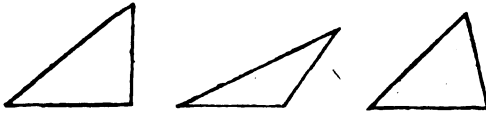An isosceles triangle, is that which has only two sides equal.

XXVI.

XXVI.
A fcalene triangle, is that which has three unequal fides.
XXVII.
A right angled triangle, is that which has a right angle.
XXVIII.
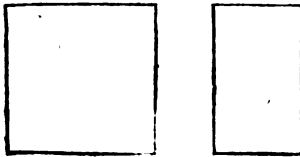An obtufe angled triangle, is that which has an obtufe angle.

XXIX.
An acute angled triangle, is that which has three acute angles.
XXX.
Of four fided figures, a fquare is that which has all its fides
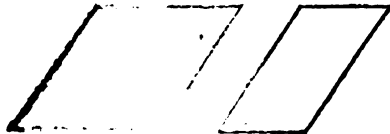equal, and all its angles right angles.

XXXI.
An oblong, is that which has all its angles right angles, but
has not all its fides equal.
XXXII.
A rhombus, is that which has all its fides equal, but its angles
are not right angles.

I.
See N. A rhomboid, is that which has its oppofite fides equal to one
another, but all its fides are not equal, nor its angles right
angles.

XXXIV.
All other four fided figures befides thefe, are called Trapeziums.

XXXV.
Parallel ftraight lines, are fuch as are in the fame plane, and which, being produced ever fo far both ways, do not meet.

————————————

POSTULATES.

I.
LET it be granted that a ftraight line may be drawn from any one point to any other point.

II.
That a terminated ftraight line may be produced to any length in a ftraight line.

III.
And that a circle may be defcribed from any centre, at any diftance from that centre.

AXIOMS.

I.
THINGS which are equal to the fame are equal to one another.

II.
If equals be added to equals, the wholes are equal.

III.
If equals be taken from equals, the remainders are equal.

IV.
If equals be added to unequals, the wholes are unequal.

V.
If equals be taken from unequals, the remainders are unequal.

VI.
Things which are double of the fame, are equal to one another.

VII.
Things which are halves of the fame, are equal to one another.

VIII.
Magnitudes which coincide with one another, that is, which exactly fill the fame fpace, are equal to one another.

IX.

IX.

The whole is greater than its part.

X.

Two ftraight lines cannot inclofe a fpace.

XI.

All right angles are equal to one another.

XII.

" If a ftraight line meets two ftraight lines, fo as to make the
" two interior angles on the fame fide of it taken together
" lefs than two right angles, thefe ftraight lines being con-
" tinually produced, fhall at length meet upon that fide on
" which are the angles which are lefs than two right angles.
" See the notes on Prop. 29. of Book I."

PROPO-

PROPOSITION I. PROBLEM.

TO defcribe an equilateral triangle upon a given fi-
nite ftraight line.

Let AB be the given ftraight line ; it is required to defcribe
an equilateral triangle upon it.

From the centre A, at the di-
ftance AB, defcribe a the circle
BCD, and from the centre B, at
the diftance BA, defcribe the
circle ACE ; and from the point
C, in which the circles cut one
another, draw the ftraight lines b
CA, CB to the points A, B ; ABC
fhall be an equilateral triangle.

a. 3. Poftu-
late.

b. 2d. Poft.
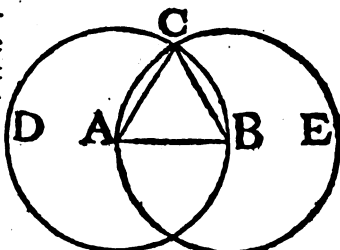
Becaufe the point A is the centre of the circle BCD, AC is
equal c to AB ; and becaufe the point B is the centre of the
circle ACE, BC is equal to BA : But it has been proved that CA
is equal to AB ; therefore CA, CB are each of them equal to
AB ; but things which are equal to the fame are equal to one
another d ; therefore CA is equal to CB ; wherefore CA, AB,
BC are equal to one another ; and the triangle ABC is there-
fore equilateral, and it is defcribed upon the given ftraight line
AB.   Which was required to be done.

c. 15th De-
finition.

d. 1ft Axi-
om.

PROP. II. PROB.

FROM a given point to draw a ftraight line equal to
a given ftraight line.

Let A be the given point, and BC the given ftraight line; it is
required to draw from the point A a ftraight line equal to BC.

From the point A to B draw a
the ftraight line AB ; and upon it
defcribe b the equilateral triangle
DAB, and produce c the ftraight
lines DA, DB, to E and F ; from
the centre B, at the diftance BC,
defcribe d the circle CGH, and
from the centre D, at the diftance
DG, defcribe the circle GKL. AL
fhall be equal to BC.
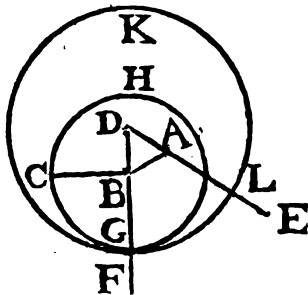
a. 1. Poft.

b. 1. 1.

c. 2. Poft.

d. 3. Poft.

Becaufe

Book I. Becaufe the point B is the centre of the circle CGH, BC is
equal ᵃ to BG; and becaufe D is the centre of the circle GKL,
a. 15. Def. DL is equal to DG, and DA, DB, parts of them, are equal;
b. 3. Ax. therefore the remainder AL is equal to the remainder ᶠ BG:
But it has been fhewn, that BC is equal to BG; wherefore AL
and BC are each of them equal to BG; and things that are
equal to the fame are equal to one another; therefore the
ftraight line AL is equal to BC. Wherefore from the given
point A a ftraight line AL has been drawn equal to the given
ftraight line BC. Which was to be done.

PROP. III. PROB.

FROM the greater of two given ftraight lines to cut
off a part equal to the lefs.

Let AB and C be the two gi-
ven ftraight lines, whereof AB is
the greater. It is required to cut
off from AB, the greater, a part
equal to C the lefs.

a. 2. 1.   From the point A draw ᵃ the
ftraight line AD equal to C;
and from the centre A, and at
b. 3. Poft. the diftance AD, defcribe ᵇ the
circle DEF; and becaufe A is
the centre of the circle DEF, AE fhall be equal to AD; but the
ftraight line C is likewife equal to AD; whence AE and C are
each of them equal to AD; wherefore the ftraight line AE is
c. 1. Ax. equal to ᶜ C, and from AB, the greater of two ftraight lines,
a part AE has been cut off equal to C the lefs. Which was to
be done.
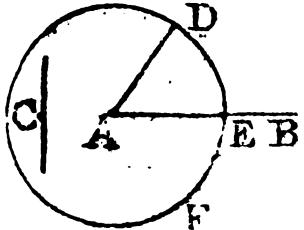
PROP. IV. THEOREM.

IF two triangles have two fides of the one equal to two
fides of the other, each to each; and have likewife
the angles contained by thofe fides equal to one another;
they fhall likewife have their bafes, or *third fides*, equal;
and the two triangles fhall be equal; and their other
angles fhall be equal, each to each, viz. thofe to which
the equal fides are oppofite.

Let ABC, DEF be two triangles which have the two fides
AB, AC equal to the two fides DE, DF, each to each, viz.
                                                            AB

AB to DE, and AC to DF; and the angle BAC equal to the angle EDF, the bafe BC fhall be equal to the bafe EF ; and the triangle ABC to the triangle DEF ; and the other angles, to which the equal fides are oppofite, fhall be equal each to each, viz. the angle ABC to the angle DEF, and the angle ACB to DFE.
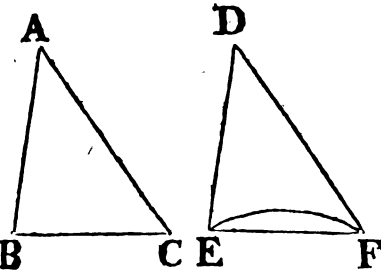
For, if the triangle ABC be applied to DEF, fo that the point A may be on D, and the ftraight line AB upon DE; the point B fhall coincide with the point E, becaufe AB is equal to DE; and AB coinciding with DE, AC fhall coincide with DF, be-caufe the angle BAC is equal to the angle EDF ; wherefore alfo the point C fhall coincide with the point F, becaufe the ftraight line AC is equal to DF : But the point B coincides with the point E ; wherefore the bafe BC fhall coincide with the bafe EF, becaufe the point B coinciding with E, and C with F, if the bafe BC does not coincide with the bafe EF, two ftraight lines would inclofe a fpace, which is impoffible [a]. Therefore the bafe BC fhall coincide with the bafe EF, and be equal to it. Wherefore the whole triangle ABC fhall coincide with the whole triangle DEF, and be equal to it ; and the other angles of the one fhall coincide with the remaining angles of the other, and be equal to them, viz. the angle ABC to the angle DEF, and the angle ACB to DFE. Therefore, if two tri-angles have two fides of the one equal to two fides of the other, each to each, and have likewife the angles contained by thofe fides equal to one another, their bafes fhall likewife be equal, and the triangles be equal, and their other angles to which the equal fides are oppofite fhall be equal, each to each. Which was to be demonftrated.

[a] 10. Ax.

P R O P. V. T H E O R.

THE angles at the bafe of an Ifofceles triangle are equal to one another ; and, if the equal fides be produced, the angles upon the other fide of the bafe fhall be equal.

Let ABC be an Ifofceles triangle, of which the fide AB is e-

B   qual

Book I. qual to AC, and let the ſtraight lines AB, AC be produced to
D and E, the angle ABC ſhall be equal to the angle ACB, and
the angle CBD to the angle BCE.

In BD take any point F, and from AE, the greater, cut off
a 3. 1. AG equal[a] to AF, the leſs, and join FC, GB.

Becauſe AF is equal to AG, and AB to AC, the two ſides
FA, AC are equal to the two GA, AB, each to each; and
they contain the angle FAG com-
mon to the two triangles AFC,
AGB; therefore the baſe FC is e-
b 4. 1. qual[b] to the baſe GB, and the tri-
angle AFC to the triangle AGB;
and the remaining angles of the one
are equal[b] to the remaining angles
of the other, each to each, to which
the equal ſides are oppoſite; viz.
the angle ACF to the angle ABG,
and the angle AFC to the angle
AGB: And becauſe the whole AF is
equal to the whole AG, of which the
parts AB, AC, are equal; the re-
c 3. Ax. mainder BF ſhall be equal[c] to the remainder CG; and FC was
proved to be equal to GB; therefore the two ſides BF, FC are
equal to the two CG, GB, each to each; and the angle BFC is
equal to the angle CGB, and the baſe BC is common to the two
triangles BFC, CGB; wherefore the triangles are equal[b],
and their remaining angles, each to each, to which the equal
ſides are oppoſite; therefore the angle FBC is equal to the angle
GCB, and the angle BCF to the angle CBG: And, ſince it has
been demonſtrated, that the whole angle ABG is equal to the
whole ACF, the parts of which, the angles CBG, BCF are alſo
equal; the remaining angle ABC is therefore equal to the re-
maining angle ACB, which are the angles at the baſe of the
triangle ABC: And it has alſo been proved that the angle FBC
is equal to the angle GCB, which are the angles upon the o-
ther ſide of the baſe Therefore the angles at the baſe, &c.
Q. E. D.

COROLLARY. Hence every equilateral triangle is alſo equi-
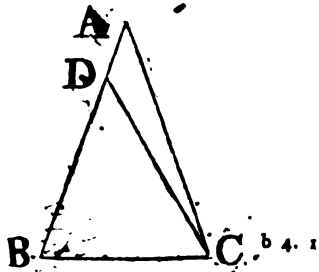angular.

PROP. VI. THEOR.

IF two angles of a triangle be equal to one another,
the ſides alſo, which ſubtend, *or are oppoſite to,* the
equal angles, ſhall be equal to one another.

Let

Let ABC be a triangle having the angle ABC equal to the
angle ACB ; the fide AB is alfo equal to the fide AC.

For, if AB be not equal to AC, one of them is greater than
the other: Let AB be the greater, and from it cut ª off DB e- a 3. 1.
qual to AC, the lefs, and join DC; there-
fore, becaufe in the triangles DBC, ACB,
DB is equal to AC, and BC common to
both, the two fides DB, BC are equal to
the two AC, CB, each to each ; and the
angle DBC is equal to the angle ACB ;
therefore the bafe DC is equal to the
bafe AB, and the triangle DBC is equal
to the triangle b ACB, the lefs to the
greater ; which is abfurd. Therefore
AB is not unequal to AC, that is, it is
equal to it. Wherefore, if two angles, &c. Q E. D.

b 4. 1.

Cor. Hence every equiangular triangle is alfo equilateral.

PROP. VII. THEOR.

UPON the fame bafe, and on the fame fide of it, See N.
there cannot be two triangles that have their fides
which are terminated in one extremity of the bafe equal
to one another, and likewife thofe which are terminated
in the other extremity.

If it be poffible, let there be two triangles ACB, ADB, up-
on the fame bafe AB, and upon the fame fide of it, which have
their fides CA, DA, terminated in the extremity A of the
bafe, equal to one another, and like-
wife their fides CB, DB that are ter-
minated in B.

Join CD ; then, in the cafe in
which the vertex of each of the tri-
angles is without the other triangle,
becaufe AC is equal to AD, the angle   a 5. 1.
ACD is equal ª to the angle ADC :
But the angle ACD is greater than
the angle BCD ; therefore the angle
ADC is greater alfo than BCD ;
much more then is the angle BDC greater than the angle BCD.
Again, becaufe CB is equal to DB, the angle BDC is equal ª to
the angle BCD ; but it has been demonftrated to be greater
than it ; which is impoffible.

B 2

But,
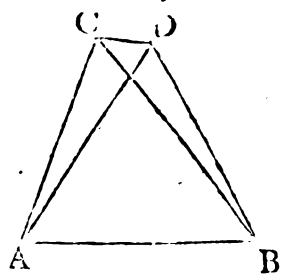
But, if one of the vertices, as D, be within the other triangle ACB; produce AC, AD to E, F; therefore, becaufe AC is equal to AD in the triangle ACD, the angles ECD, FDC upon the other fide of the bafe CD are equal[a] to one another, but the angle ECD is greater than the angle BCD; wherefore the angle FDC is likewife greater than BCD; much more then is the angle BDC greater than the angle BCD. Again, becaufe CB is equal to DB, the angle BDC is equal[a] to the angle BCD; but BDC has been proved to be greater than the fame BCD; which is impoffible. The cafe in which the vertex of one triangle is upon a fide of the other, needs no demonftration.
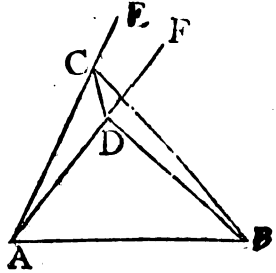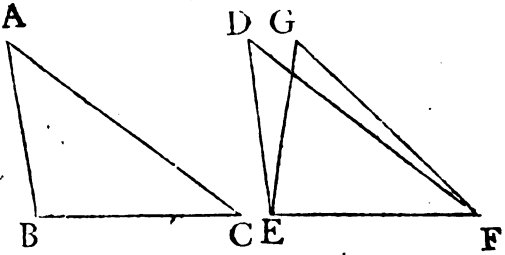
Therefore upon the fame bafe, and on the fame fide of it, there cannot be two triangles that have their fides which are terminated in one extremity of the bafe equal to one another, and likewife thofe which are terminated in the other extremity. Q. E. D.

PROP. VIII THEOR.

IF two triangles have two fides of the one equal to two fides of the other, each to each, and have likewife their bafes equal; the angle which is contained by the two fides, of the one fhall be equal to the angle contained by the two fides equal to them, of the other.

Let ABC, DEF be two triangles having the two fides AB, AC equal to the two fides DE, DF, each to each, viz. AB to DE, and AC to DF; and alfo the bafe BC equal to the bafe EF. The angle BAC is equai to the angle EDF.

For, if the triangle ABC be applied to DEF, fo that the point B be on E, and the ftraight line BC upon EF, the point C fhall alfo coincide with the point F, Becaufe BC

BC is equal to EF; therefore BC coinciding with EF, BA and
AC fhall coincide with ED and DF; for, if the bafe BC coin-
cides with the bafe EF, but the fides BA, CA do not coin-
cide with the fides ED, FD, but have a different fituation, as
EG, FG; then, upon the fame bafe EF, and upon the fame
fide of it, there can be two triangles that have their fides
which are terminated in one extremity of the bafe equal to one
another, and likewife their fides terminated in the other extre-
mity: But this is impoffible ᵃ; therefore, if the bafe BC coin-   ᵃ 7. 1.
cides with the bafe EF, the fides BA, AC cannot but coincide
with the fides ED, DF; wherefore likewife the angle BAC
coincides with the angle EDF, and is equal ᵇ to it. There-   ᵇ 8. Ax·
fore if two triangles, &c.   Q. E. D.

P R O P. IX.   P R O B.

T O bifect a given rectilineal angle, that is, to divide
it into two equal angles.

Let **BAC** be the given rectilineal angle, it is required to bi-
fect it.

Take any point D in AB, and from AC cut ᵃ off AE e- ᵃ 3. 1.
qual to AD; join DE, and upon it de-
fcribe ᵇ an equilateral triangle DEF;   ᵇ 1. 1.
then join AF; the ftraight line AF
bifects the angle BAC.

Becaufe AD is equal to AE, and
AF is common to the two triangles
DAF, EAF; the two fides DA, AF,
are equal to the two fides EA, AF,
each to each; and the bafe DF is e-
qual to the bafe EF; therefore the
angle DAF is equal ᶜ to the angle   ᶜ 8. 1.
EAF; wherefore the given rectilineal angle BAC is bifected
by the ftraight line AF.   Which was to be done.
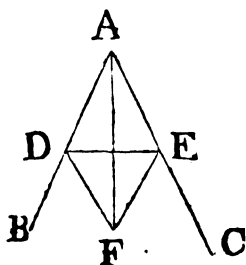
P R O P. X.   P R O B.

T O bifect a given finite ftraight line, that is, to divide
it into two equal parts.

Let **AB** be the given ftraight line; it is required to divide it
into two equal parts.

Defcribe ᵃ upon it an equilateral triangle ABC, and bifect ᵃ 1. 1.
ᵇ the angle ACB by the ftraight line CD. AB is cut into two ᵇ 9. 1.
equal parts in the point D.

Becaufe

Becaufe AC is equal to CB, and CD common to the two triangles ACD, BCD ; the two fides AC, CD are e- qual to BC, CD, each to each ; and the angle ACD is equal to the angle BCD ; therefore the bafe AD is equal to the

e 4. 1.

bafe c DB, and the ftraight line AB is divided into two equal parts in the point D. Which was to be done.
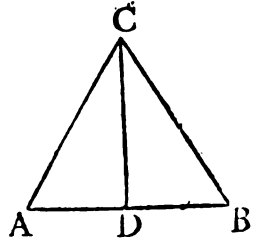
PROP. XI. PROB.

TO draw a ftraight line at right angles to a given ftraight line, from a given point in the fame.

See N.

Let AB be a given ftraight line, and C a point given in it ; it is required to draw a ftraight line from the point C at right angles to AB.

a 3. 1.

b 1. 1.

Take any point D in AC, and a make CE equal to CD, and upon DE defcribe b the equila- teral triangle DFE, and join FC ; the ftraight line FC drawn from the given point C is at right angles to the given ftraight line AB.
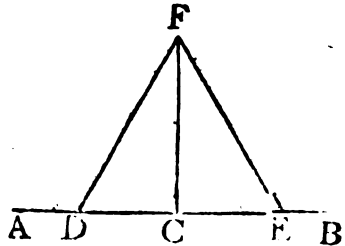
Becaufe DC is equal to CE, and FC common to the two triangles DCF, ECF ; the two fides DC, CF, are equal to the

c 8. 1.

two EC, CF, each to each ; and the bafe DF is equal to the bafe EF ; therefore the angle DCF is equal c to the angle ECF ; and they are adjacent angles. But, when the adjacent angles which one ftraight line makes with another ftraight line are

d 10. Def.

1.

equal to one another, each of them is called a right d angle ; therefore each of the angles DCF, ECF, is a right angle. Wherefore, from the given point C, in the given ftraight line AB, FC has been drawn at right angles to AB. Which was to be done.

COR. By help of this problem, it may be demonftrated, that two ftraight lines cannot have a common fegment.

If it be poffible, let the two ftraight lines ABC, ABD have the fegment AB common to both of them. From the point B draw BE at right angles to AB ; and becaufe ABC is a ftraight line,

Book I.
ᵃ 10. Def.
I.

line, the angle CBE is equal ᵃ to the angle EBA; in the same manner, because ABD is a straight line, the angle DBE is equal to the angle EBA; wherefore the angle DBE is equal to the angle CBE, the less to the greater; which is impossible; therefore two straight lines cannot have a common segment.
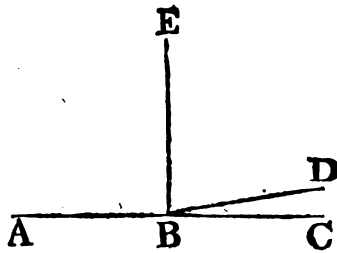
P R O P. XII. P R O B.

TO draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced to any length both ways, and let C be a point without it. It is required to draw a straight line perpendicular to AB from the point C.

Take any point D upon the other side of AB, and from the centre C, at the distance CD, describe ᵇ the circle EGF meeting AB in F, G; and bisect ᶜ FG in H, and join CF, CH, CG; the straight line CH, drawn from the given point C, is perpendicular to the given straight line AB.
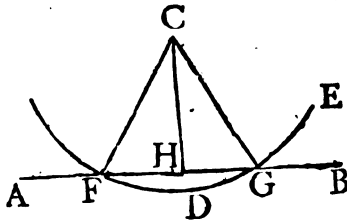
ᵇ 3. Post.

ᶜ 10. 1.

Because FH is equal to HG, and HC common to the two triangles FHC, GHC, the two sides FH, HC are equal to the two GH, HC, each to each; and the base CF is equal ᵈ to the base CG; therefore the angle CHF is equal ᵉ to the angle CHG; and they are adjacent angles; but when a straight line standing on a straight line makes the adjacent angles equal to one another, each of them is a right angle, and the straight line which stands upon the other is called a perpendicular to it; therefore from the given point C a perpendicular CH has been drawn to the given straight line AB.   Which was to be done.

ᵈ 15. Def.
I.
ᵉ 8. I.

P R O P. XIII. T H E O R.

THE angles which one straight line makes with another upon the one side of it, are either two right angles, or are together equal to two right angles.

B 4

Let

Let the ftraight line AB make with CD, upon one fide of it, the angles CBA, ABD ; thefe are either two right angles, or are together equal to two right angles

For, if the angle CBA be equal to ABD, each of them is a
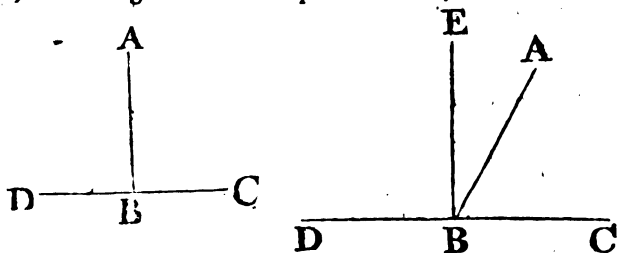
a def. 10.   right ᵃ angle ; but, if not, from the point B draw BE at right
b 11. 1.   angles ᵇ to CD ; therefore the angles CBE, EBD are two right
             angles ₐ; and becaufe CBE is equal to the two angles CBA, ABE
             together, add the angle EBD to each of thefe equals ; there-
c 2. Ax.   fore the angles CBE, EBD are equal ᶜ to the three angles CBA,
             ABE, EBD. Again, becaufe the angle DBA is equal to the
             two angles DBE, EBA, add to thefe equals the angle ABC ;
             therefore the angles DBA, ABC are equal to the three angles
             DBE, EBA, ABC ; but the angles CBE, EBD have been de-
             monftrated to be equal to the fame three angles ; and things
d 1. Ax.   that are equal to the fame are equal ᵈ to one another; therefore
             the angles CBE, EBD are equal to the angles DBA, ABC ; but
             CBE, EBD are two right angles ; therefore DBA, ABC are
             together equal to two right angles. Wherefore, when a ftraight
             line, &c. Q. E. D.

PROP. XIV. THEOR.

IF, at a point in a ftraight line, two other ftraight lines, upon the oppofite fides of it, make the adjacent angles together equal to two right angles, thefe two ftraight lines fhall be in one and the fame ftraight line.

At the point B in the ftraight line AB, let the two ftraight lines BC, BD upon the oppofite fides of AB, make the adjacent angles ABC, ABD equal together to two right angles. BD is in the fame ftraight line with CB.

For, if BD be not in the fame ftraight line with CB, let BE be

in
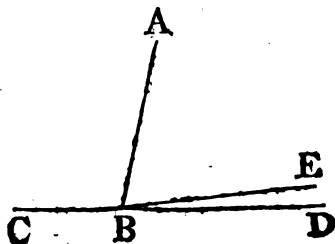
in the fame ſtraight line with it; therefore, becauſe the ſtraight line AB makes angles with the ſtraight line CBE, upon one ſide of it, the angles ABC, ABE are together equal ᵃ to two ᵃ 13. 1. right angles; but the angles ABC, ABD are likewiſe together equal to two right angles; therefore the angles CBA, ABE are equal to the angles CBA, ABD: Take away the common angle ABC, the remaining angle ABE is equal ᵇ to the remaining ᵇ 3. Ax. angle ABD, the leſs to the greater, which is impoſſible; there-fore BE is not in the ſame ſtraight line with BC. And, in like manner, it may be demonſtrated, that no other can be in the ſame ſtraight line with it but BD, which therefore is in the ſame ſtraight line with CB. Wherefore, if at a point, &c. Q. E. D.

PROP. XV. THEOR.

IF two ſtraight lines cut one another, the vertical, *or op-poſite*, angles ſhall be equal.

Let the two ſtraight lines AB, CD cut one another in the point E; the angle AEC ſhall be equal to the angle DEB, and CEB to AED.

Becauſe the ſtraight line AE makes with CD the angles CEA, AED, theſe angles are toge-ther equal ᵃ to two right angles. Again, becauſe the ſtraight line DE makes with AB the angles AED, DEB, theſe alſo are together equal ᵃ to two right angles; and CEA, AED have been demonſtrated to be equal to two right angles; wherefore the angles CEA, AED are equal to the angles AED, DEB. Take away the common angle AED, and the remaining angle CEA is equal ᵇ to the remain- ᵇ 3. Ax. ing angle DEB. In the ſame manner it can be demonſtrated that the angles CEB, AED are equal. Therefore, if two ſtraight lines, &c. Q. E. D.
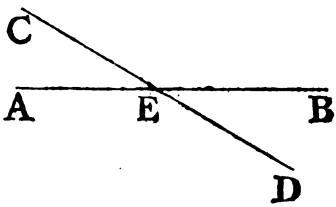
ᵃ 13. 1.

COR. 1. From this it is manifeſt, that, if two ſtraight lines cut one another, the angles they make at the point where they cut, are together equal to four right angles.

COR. 2. And conſequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROP.

P R O P. XVI.　T H E O R.

IF one fide of a triangle be produced, the exterior angle is greater than either of the interior oppofite angles.
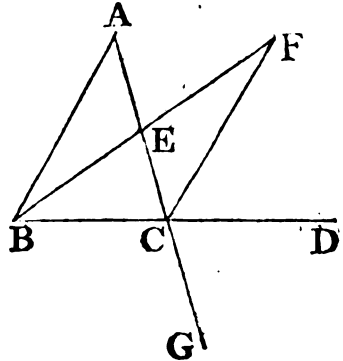
Let ABC be a triangle, and let its fide BC be produced to D, the exterior angle ACD is greater than either of the interior oppofite angles CBA, BAC.

a. 10. 1.　Bifect a AC in E, join BE and produce it to F, and make EF equal to BE; join alfo FC, and produce AC to G.

Becaufe AE is equal to EC, and BE to EF; AE, EB are equal to CE, EF, each to each; and the angle AEB is
b. 15. 1.　equal b to the angle CEF, becaufe they are oppofite vertical angles; therefore the
c. 4 1.　bafe AB is equal c to the bafe CF, and the triangle AEB to the triangle CEF, and the remaining angles to the remaining angles, each to each, to which the equal fides are oppofite; wherefore the angle BAE is equal to the angle ECF; but the angle ECD is greater than the angle ECF; therefore the angle ACD is greater than BAE : In the fame manner, if the fide BC be bifected, it may be demonftrated
d. 15. 1.　that the angle BCG, that is d, the angle ACD, is greater than the angle ABC. Therefore, if one fide, &c. Q. E. D.
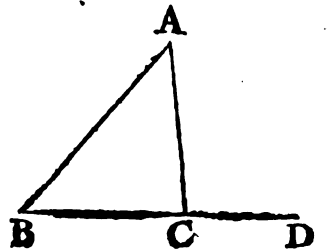
P R O P. XVII.　T H E O R.

ANY two angles of a triangle are together lefs than two right angles.

Let ABC be any triangle; any two of its angles together are lefs than two right angles.

Produce BC to D; and becaufe ACD is the exterior angle of the triangle ABC, ACD is
a. 16. 1.　greater a than the interior and oppofite angle ABC; to each of

thefe

thefe add the angle ACB; therefore the angles ACD, ACB are greater than the angles ABC, ACB; but ACD, ACB are together equal [b] to two right angles; therefore the angles ABC, BCA are lefs than two right angles. In like manner, it may be demonftrated, that BAC, ACB, as alfo CAB, ABC are lefs than two right angles. Therefore any two angles, &c. Q. E. D.

P R O P. XVIII. T H E O R.

THE greater fide of every triangle is oppofite to the greater angle.

Let ABC be a triangle, of which the fide AC is greater than the fide AB; the angle ABC is alfo greater than the angle BCA.

Becaufe AC is greater than AB, make [a] AD equal to AB, and join BD; and becaufe ADB is the exterior angle of the triangle BDC, it is greater [b] than the interior and oppofite angle DCB; but ADB is equal [c] to ABD, becaufe the fide AB is equal to the fide AD; therefore the angle ABD is likewife greater than the angle ACB; wherefore much more is the angle ABC greater than ACB. Therefore the greater fide, &c. Q. E. D.
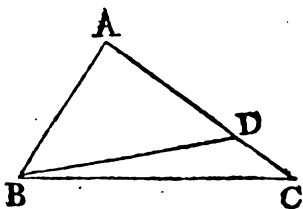
P R O P. XIX. T H E O R.

THE greater angle of every triangle is fubtended by the greater fide, or *has the greater fide oppofite to it.*

Let ABC be a triangle, of which the angle ABC is greater than the angle BCA; the fide AC is likewife greater than the fide AB.

For, if it be not greater, AC muft either be equal to AB, or lefs than it; it is not equal, becaufe then the angle ABC would be equal [a] to the angle ACB; but it is not; therefore AC is not equal to AB; neither is it lefs; becaufe then the angle

ABC
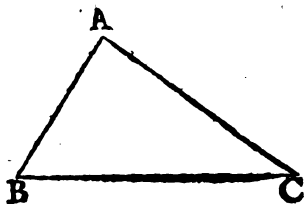
ABC would be lefs ᵇ than the angle ACB; but it is not; there-
fore the fide AC is not lefs than **AB**; and it has been ſhewn
that it is not equal to **AB**; therefore AC is greater than **AB**.
Wherefore the greater angle, &c. Q. E. D.

PROP. XX.   THEOR.

ANY two fides of a triangle are together greater than
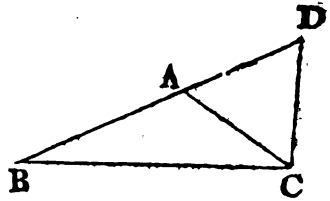the third fide.

Let ABC be a triangle; any two fides of it together are
greater than the third fide, viz. the fides BA, AC greater than
the fide BC; and AB, BC greater than AC; and BC, CA
greater than AB.

Produce BA to the point D,
and make ᵃ AD equal to AC;
and join DC.

Becauſe DA is equal to AC,
the angle ADC is likewife equal
ᵇ to ACD; but the angle BCD
is greater than the angle ACD;
therefore the angle BCD is great-
er than the angle ADC; and be-
cauſe the angle BCD of the triangle DCB is greater than its

angle BDC, and that the greater ᶜ fide is oppofite to the greater
angle; therefore the fide DB is greater than the fide BC; but
DB is equal to BA and AC; therefore the fides BA, AC are
greater than BC.   In the fame manner it may be demonſtrated,
that the fides AB, BC are greater than CA, and BC, CA
greater than AB.   Therefore any two fides, &c. Q. E. D.

PROP. XXI.   THEOR.

IF, from the ends of the fide of a triangle, there be
drawn two ſtraight lines to a point within the triangle,
thefe ſhall be lefs than the other two fides of the tri-
angle, but ſhall contain a greater angle.

Let the two ſtraight lines BD, CD be drawn from B, C,
the ends of the fide BC of the triangle ABC, to the point D
within it; BD and DC are lefs than the other two fides BA,
AC of the triangle, but contain an angle BDC greater than the
angle BAC.

Droduce BD to E; and becauſe two fides of a triangle are
greater than the third fide, the two fides BA, AE of the tri-
angle

angle ABE are greater than BE. To each of thefe add EC;
therefore the fides BA, AC are
greater than BE, EC: Again,
becaufe the two fides CE, ED
of the triangle CED are great-
er than CD, add DB to each
of thefe; therefore the fides
CE, EB are greater than CD,
DB; but it has been fhewn that
BA, AC are greater than BE,
EC; much more then are BA
AC greater than BD, DC.
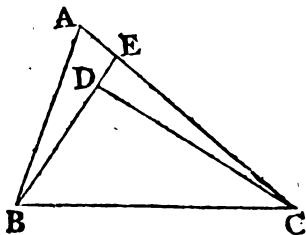
Again, becaufe the exterior angle of a triangle is greater than
the interior and oppofite angle, the exterior angle BDC of the
triangle CDE is greater than CED; for the fame reafon, the ex-
terior angle CEB of the triangle ABE is greater than BAC; and
it has been demonftrated that the angle BDC is greater than
the angle CEB; much more then is the angle BDC greater than
the angle BAC. Therefore, if from the ends of, &c. Q. E. D.

P R O P. XXII.   P R O B.

TO make a triangle of which the fides fhall be equal See N.
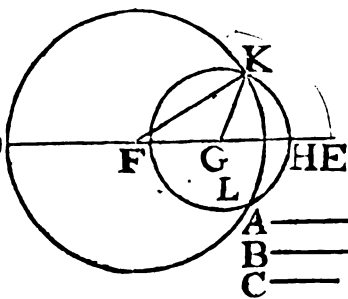to three given ftraight lines, but any two whatever
of thefe muft be greater than the third[a].

Let A, B, C be the three given ftraight lines, of which any
two whatever are greater than the third, viz. A and B greater
than C; A and C greater than B; and B and C than A. It is
required to make a triangle of which the fides fhall be equal to
A, B, C, each to each.

Take a ftraight line DE terminated at the point D, but un-
limited towards E, and
make[a] DF equal to A, FG
to B, and GH equal to C;
and from the centre F, at
the diftance FD, defcribe
[b] the circle DKL; and
from the centre G, at the
diftance GH, defcribe[b]
another circle HLK, and
join KF, KG; the tri-
angle KFG has its fides
equal to the three ftraight
lines, A, B, C.

Becaufe the point F is the centre of the circle DKL, FD is
equal

a 20. I.

a 3. I.

b 3. Poft.

equal ᶜ to FK; but FD is equal to the ſtraight line A; there-
fore FK is equal to A: Again, becauſe G is the centre of the
circle LKH, GH is equal ᶜ to GK; but GH is equal to C;
therefore alſo GK is equal to C; and FG is equal to B; there-
fore the three ſtraight lines KF, FG, GK are equal to the three
A, B, C: And therefore the triangle KFG has its three ſides
KF, FG, GK equal to the three given ſtraight lines A, B, C.
Which was to be done.

P R O P. XXIII.   P R O B.

A T a given point in a given ſtraight line, to make a
rectilineal angle equal to a given rectilineal angle.

Let AB be the given ſtraight line, and A the given point in it,
and DCE the given rectilineal angle; it is required to make an
angle at the givenpoint
A in the given ſtraight
line AB, that ſhall be
equal to the given rec-
tilineal angle DCE.

    Take in CD, CE, any
points D, E, and join
DE; and make a the
triangle AFG the ſides
of which ſhall be equal
to the three ſtraight
lines CD, DE, CE, ſo
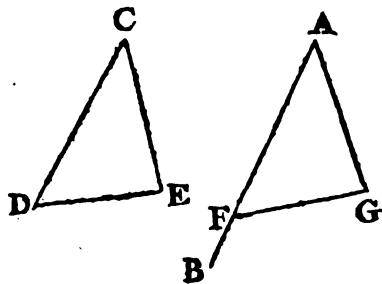
that CD be equal to AF, CE to AG, and DE to FG; and becauſe
DC, CE are equal to FA, AG, each to each, and the baſe DE
to the baſe FG; the angle DCE is equal ᵇ to the angle FAG.
Therefore, at the given point A in the given ſtraight line AB,
the angle FAG is made equal to the given rectilineal angle
DCE. Which was to be done.

P R O P. XXIV.   T H E O R.

I F two triangles have two ſides of the one equal to two
ſides of the other, each to each, but the angle con-
tained by the two ſides of one of them greater than the
angle contained by the two ſides equal to them, of the
other; the baſe of that which has the greater angle ſhall
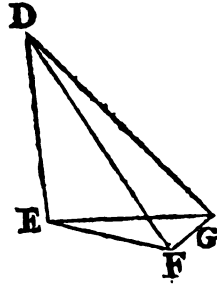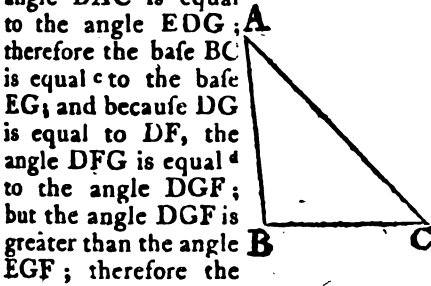be greater than the baſe of the other.

    Let ABC, DEF be two triangles which have the two ſides
AB,

AB, AC equal to the two DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF; the bafe BC is alfo greater than the bafe EF.

Of the two fides DE, DF, let DE be the fide which is not greater than the other, and at the point D, in the ftraight line DE, make ª the angle EDG equal to the angle BAC; and make DG equal ᵇ to AC or DF, and join EG, GF.

a. 23. 1.
b. 3. 1.

Becaufe AB is equal to DE, and AC to DG, the two fides BA, AC are equal to the two ED, DG, each to each, and the angle BAC is equal to the angle EDG; therefore the bafe BC is equal ᶜ to the bafe EG; and becaufe DG is equal to DF, the angle DFG is equal ᵈ to the angle DGF; but the angle DGF is greater than the angle EGF; therefore the

c. 4. 1.

d. 5. 1.

angle DFG is greater than EGF; and much more is the angle EFG greater than the angle EGF; and becaufe the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater ᵉ fide is oppofite to the greater angle; the fide EG is therefore greater than the fide EF; but EG is equal to BC; and therefore alfo BC is greater than EF. Therefore, if two triangles, &c. Q. E. D.

e. 19. 1.

P R O P. XXV. T H E O R.

IF two triangles have two fides of the one equal to two fides of the other, each to each, but the bafe of the one greater than the bafe of the other; the angle alfo contained by the fides of that which has the greater bafe, fhall be greater than the angle contained by the fides equal to them, of the other.

Let ABC, DEF be two triangles which have the two fides AB, AC equal to the two fides DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the bafe CB is greater than the bafe EF; the angle BAC is likewife greater than the angle EDF.
For,

For, if it be not greater, it muſt either be equal to it, or leſs; but the angle BAC is not equal to the angle EDF, be-

cauſe then the baſe
BC would be equal [a]
to EF; but it is not;
therefore the angle
BAC is not equal to
the angle EDF; nei-
ther is it leſs; becauſe
then the baſe BC
would be leſs [b] than
the baſe EF; but it
is not; therefore the

angle BAC is not leſs than the angle EDF; and it was ſhewn that it is not equal to it; therefore the angle BAC is greater than the angle EDF. Wherefore, if two triangles, &c. Q. E. D.
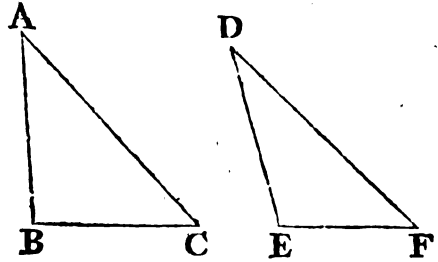
PROP. XXVI. THEOR.

IF two triangles have two angles of one equal to two angles of the other, each to each; and one ſide e-qual to one ſide, viz. either the ſides adjacent to the equal angles, or the ſides oppoſite to equal angles in each; then ſhall the other ſides be equal, each to each; and alſo the third angle of the one to the third angle of the other.

Let ABC, DEF be two triangles which have the angles ABC, BCA equal to the angles DEF, EFD, viz. ABC to DEF, and BCA to EFD; alſo one ſide equal to one ſide; and firſt let thoſe ſides be equal which are adjacent to the angles that are equal in the two triangles, viz.

BC to EF; the other
ſides ſhall be equal,
each to each, viz. AB
to DE, and AC to DF;
and the third angle
BAC to the third
angle EDF.
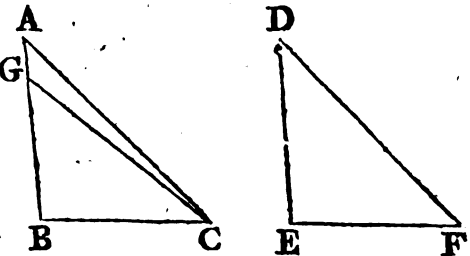
For, if AB be not
equal to DE, one of
them muſt be the
greater. Let AB be the greater of the two, and make BG equal to DE, and join GC; therefore, becauſe BG is equal

equal to DE, and BC to EF, the two fides GB, BC, are equal
to the two DE, EF, each to each ; and the angle GBC is equal
to the angle DEF ; therefore the bafe GC is equal ᵃ to the bafe   ᵃ 4. 1.
DF, and the triangle GBC to the triangle DEF, and the other
angles to the other angles, each to each, to which the equal
fides are oppofite ; therefore the angle GCB is equal to the
angle DFE ; but DFE is, by the hypothefis, equal to the angle
BCA ; wherefore alfo the angle BCG is equal to the angle
BCA, the lefs to the greater, which is impoffible; therefore
AB is not unequal to DE, that is, it is equal to it ; and BC is
equal to EF ; therefore the two AB, BC are equal to the two
DE, EF, each to each ; and the angle ABC is equal to the angle
DEF ; the bafe therefore AC is equal ᵃ to the bafe DF, and
the third angle BAC to the third angle EDF.

Next, let the fides
which are oppofite to
equal angles in each
triangle be equal to
one another, viz. AB
to DE ; likewife in
this cafe, the other
fides fhall be equal,
AC to DF, and BC
to EF ; and alfo the
third angle BAC to the third EDF.
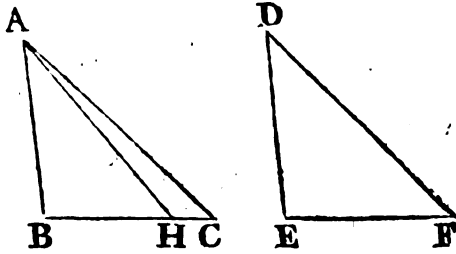
For, if BC be not equal to EF, let BC be the greater of them,
and make BH equal to EF, and join AH ; and becaufe BH is
equal to EF, and AB to DE ; the two AB, BH are equal to
the two DE, EF, each to each ; and they contain equal angles ;
therefore the bafe AH is equal to the bafe DF, and the triangle
ABH to the triangle DEF, and the other angles fhall be equal,
each to each, to which the equal fides are oppofite ; therefore
the angle BHA is equal to the angle EFD ; but EFD is equal
to the angle BCA ; therefore alfo the angle BHA is equal to
the angle BCA, that is, the exterior angle BHA of the triangle
AHC is equal to its interior and oppofite angle ECA ; which
is impoffible ᵇ ; wherefore BC is not unequal to EF, that is,   ᵇ 16. 1.
it is equal to it ; and AB is equal to DE ; therefore the two AB,
BC are equal to the two DE, EF, each to each ; and they contain
equal angles ; wherefore the bafe AC is equal to the bafe DF,
and the third angle BAC to the third angle EDF. Therefore,
if two triangles, &c. Q. E. D.
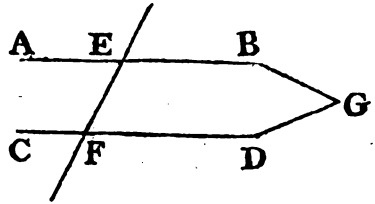
<div align="center">C</div>

PROP.

PROP. XXVII. THEOR.

IF a ſtraight line falling upon two other ſtraight lines makes the alternate angles equal to one another, theſe two ſtraight lines ſhall be parallel.

Let the ſtraight line EF, which falls upon the two ſtraight lines AB, CD, make the alternate angles AEF, EFD equal to one another; AB is parallel to CD.

For, if it be not parallel, AB and CD being produced ſhall meet either towards B, D, or towards A, C; let them be produced and meet towards B, D in the point G; therefore GEF is a tri-

a 16. 1. angle, and its exterior angle AEF is greater ᵃ than the interior and oppoſite angle EFG;
but it is alſo equal to it, which is impoſſible; there-
fore AB and CD being pro-
duced do not meet towards
B, D. In like manner it may
be demonſtrated that they do
not meet towards A, C; but
thoſe ſtraight lines which
meet neither way, though

b 35. Def. produced ever ſo far, are parallel ᵇ to one another. AB therefore is parallel to CD. Wherefore, if a ſtraight line, &c. Q. E. D.

PROP. XXVIII. THEOR.

IF a ſtraight line falling upon two other ſtraight lines makes the exterior angle equal to the interior and op-
poſite upon the ſame ſide of the line; or makes the in-
terior angles upon the ſame ſide together equal to two right angles; the two ſtraight lines ſhall be parallel to one another.

Let the ſtraight line EF, which falls upon the two ſtraight lines AB, CD, make the exterior angle EGB equal to the interior and oppoſite angle GHD upon the ſame ſide; or make the interior angles on the ſame ſide BGH, GHD together equal to two right angles; AB is parallel to CD.

Becauſe the angle EGB is e qual to the angle GHD, and the
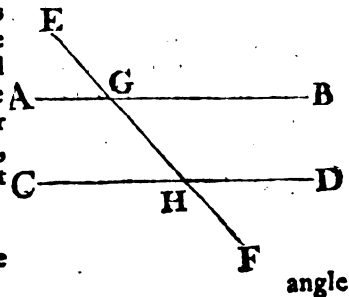
angle

angle EGB equal ᵃ to the angle AGH, the angle AGH is equal
to the angle GHD; and they are the alternate angles; therefore ᵃ 15. 1.
AB is parallel ᵇ to CD.   Again, becauſe the angles BGH, GHD ᵇ 27. 1.
are equal ᶜ to two right angles, and that AGH, BGH are alſo ᶜ By Hyp.
equal ᵈ to two right angles; the angles AGH, BGH are equal ᵈ 13. 1.
to the angles BGH, GHD: Take away the common angle
BGH, therefore the remaining angle AGH is equal to the re-
maining angle GHD; and they are alternate angles; there-
fore AB is parallel to CD.   Wherefore, if a ſtraight line, &c.
Q. E. D.

P R O P.   XXIX.   T H E O R.

IF a ſtraight line falls upon two parallel ſtraight lines, it See the
   makes the alternate angles equal to one another; and notes on
this pro-
the exterior angle equal to the interior and oppoſite upon ſition.
the ſame ſide; and likewiſe the two interior angles upon
the ſame ſide together equal to two right angles.

Let the ſtraight line EF fall upon the parallel ſtraight lines
AB, CD; the alternate angles AGH, GHD are equal to one
another; and the exterior angle EGB is equal to the interior
and oppoſite, upon the ſame ſide,
GHD;   and   the   two   interior
angles BGH, GHD upon the ſame
ſide are together equal to two right
angles.
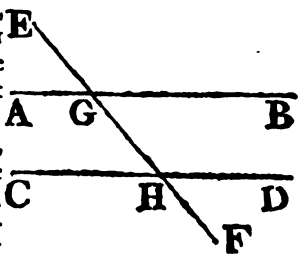
For, if AGH be not equal to GHD,
one of them muſt be greater than the
other; let AGH be the greater; and
becauſe the angle AGH is greater
than the angle GHD, add to each of
them the angle BGH; therefore the angles AGH, BGH are
greater than the angles BGH, GHD; but the angles AGH, ᵃ 13. 1.
BGH are equal ᵃ to two right angles; therefore the angles
BGH, GHD are leſs than two right angles; but thoſe ſtraight
lines which, with another ſtraight line falling upon them, make
the interior angles on the ſame ſide leſs than two right angles,
do meet ⁎ together if continually produced; therefore the ⁎ 12. ax.
ſtraight lines AB, CD, if produced far enough, ſhall meet; but See the
notes on
they never meet, ſince they are parallel by the hypotheſis; this propo-
therefore the angle AGH is not unequal to the angle GHD, ſition.
that is, it is equal to it; but the angle AGH is equal ᵇ to the ᵇ 15. 1.
angle EGB; therefore likewiſe EGB is equal to GHD; add to

c 13. 1.

each of thefe the angle BGH; therefore the angles EGB, BGH are equal to the angles BGH, GHD ; but EGB, BGH are e-qual ^{c} to two right angles; therefore alfo BGH, GHD are e-qual to two right angles. Wherefore, if a ftraight line, &c. Q. E. D.
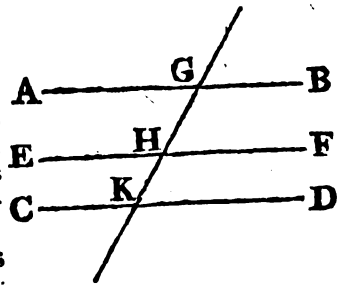
P R O P. XXX.   T H E O R.

STRAIGHT lines which are parallel to the fame ftraight line are parallel to one another.

Let AB, CD be each of them parallel to EF ; AB is alfo parallel to CD.

Let the ftraight line GHK cut AB, EF, CD ; and becaufe GHK cuts the parallel ftraight lines AB, EF, the angle AGH is equal ^{a} to the angle GHF. Again, becaufe the ftraight line GK cuts the parallel ftraight lines EF, CD, the angle GHF is equal ^{a} to the angle GKD ; and it was fhewn that the angle AGK is e-qual to the angle GHF ; there fore alfo AGK is equal to GKD; and they are alternate angles ; therefore AB is parallel ^{b} to CD. Wherefore ftraight lines, &c. Q. E. D.

a 29. 1.

b 27. 1.

P R O P. XXXI.   P R O B.

TO draw a ftraight line through a given point paral-lel to a given ftraight line.

Let A be the given point, and BC the given ftraight line ; it is required to draw a ftraight line through the point A, parallel to the ftraight line BC.
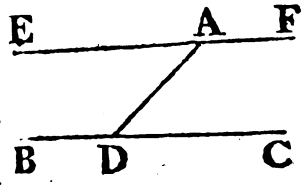
In BC take any point D, and join AD ; and at the point A in the ftraight line AD make ^{a} the angle DAE equal to the angle ADC ; and produce the ftraight line EA to F.

a 23. 1.

Becaufe the ftraight line AD, which meets the two ftraight lines BC, EF, makes the alternate angles EAD, ADC equal to one another, EF is parallel ^{b} to BC. Therefore the ftraight line

b 27. 1.

EAF

EAF is drawn through the given point A parallel to the given ſtraight line BC. Which was to be done.

PROP. XXXII. THEOR.

IF a ſide of any triangle be produced, the exterior angle is equal to the two interior and oppoſite angles ; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, and let one of its ſides BC be pro-duced to D ; the exterior angle ACD is equal to the two inte-rior and oppoſite angles CAB, ABC ; and the three interior angles of the triangle, viz. ABC, BCA, CAB are together e-qual to two right angles.

Through the point C draw CE parallel ª to the ſtraight line AB ; and becauſe AB is parallel to CE and AC meets them, the alternate angles BAC, ACE are equal ᵇ. A-gain, becauſe AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and
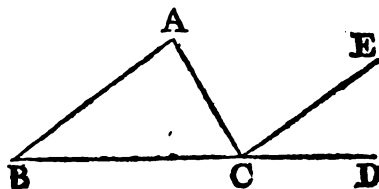
a 31. 1.

b 29. 1.

oppoſite angle ABC ; but the angle ACE was ſhown to be equal to the angle BAC ; therefore the whole exterior angle ACD is equal to the two interior and oppoſite angles CAB, ABC ; to theſe equals add the angle ACB, and the angles ACD, ACB are equal to the three angles CBA, BAC, ACB ; but the angles ACD, ACB are equal ᶜ to two right angles ; therefore alſo the angles CBA, BAC, ACB are equal to two right angles. Wherefore, if a ſide of a triangle, &c. Q. E. D.

c 13. 1.

Cor. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the ngure has ſides.

For any rectilineal figure ABCDE can be divided into as many triangles as the figure has ſides, by drawing ſtraight lines from a point F within the figure

C 3
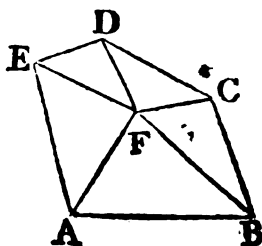
to each of its angles. And, by the preceding propofition, all the angles of thefe triangles are equal to twice as many right angles as there are triangles, that is, as there are fides of the figure ; and the fame angles are equal to the angles of the fi-gure, together with the angles at the point F, which is the com-
a 2. Cor. mon vertex of the triangles ; that is ª, together with four right
15. 1. angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has fides.

CoR. 2. All the exterior angles of any rectilineal figure, are together equal to four right angles.

Becaufe every interior angle ABC, with its adjacent exterior
b 13. 1. ABD, is equal ᵇ to two right angles; therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as there are fides of the figure; that is, by the foregoing corollary, they are equal to all the inte-rior angles of the figure, toge-ther with four right angles ; therefore all the exterior angles are equal to four right angles.

PROP. XXXIII. THEOR.

THE ftraight lines which join the extremities of two equal and parallel ftraight lines, towards the fame parts, are alfo themfelves equal and parallel.

Let AB, CD be equal and pa-rallel ftraight lines, and joined towards the fame parts by the ftraight lines AC, BD; AC, BD are alfo equal and parallel.

Join BC; and becaufe AB is pa-rallel to CD, and BC meets them, the alternate angles ABC, BCD
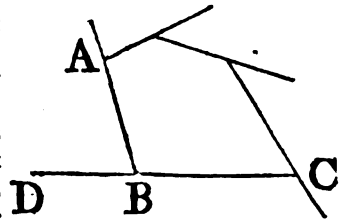**a 29 I.** are equal ª ; and becaufe AB is equal to CD, and BC common to the two triangles ABC, DCB, the two fides AB, BC are e-qual to the two DC, CB; and the angle ABC is equal to the
b 4. 1. angle BCD ; therefore the bafe AC is equal ᵇ to the bafe BD, and the triangle ABC to the triangle BCD, and the other angles

to
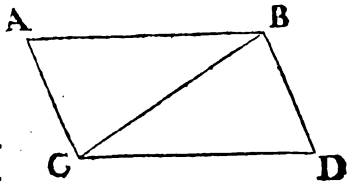
to the other angles b, each to each, to which the equal fides are
oppofite; therefore the angle ACB is equal to the angle CBD;
and becaufe the ftraight line BC meets the two ftraight lines b 4. 1.
AC, BD, and makes the alternate angles ACB, CBD equal to
one another, AC is parallel c to BD; and it was fhown to be c 27. 1.
equal to it. Therefore ftraight lines, &c. Q. E. D.

PROP. XXXIV. THEOR.

THE oppofite fides and angles of parallelograms are
equal to one another, and the diameter bifects them,
that is, divides them in two equal parts.

N. B. *A parallelogram is a four fided figure, of which
the oppofite fides are parallel; and the diameter is the
ftraight line joining two of its oppofite angles.*

Let ACDB be a parallelogram, of which BC is a diameter;
the oppofite fides and angles of the figure are equal to one an-
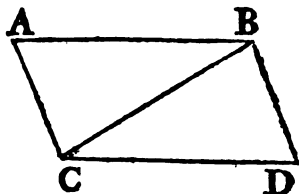other; and the diameter BC bifects it.

Becaufe AB is parallel to CD,
and BC meets them, the alter-
nate angles ABC, BCD are e-
qual a to one another; and be-
caufe AC is parallel to BD, and
BC meets them, the alternate
angles ACB, CBD are equal a

a 29. 1.

to one another; wherefore the two triangles ABC, CBD have
two angles ABC, BCA in one, equal to two angles BCD, CBD
in the other, each to each, and one fide BC common to the
two triangles, which is adjacent to their equal angles; there-
fore their other fides fhall be equal, each to each, and the
third angle of the one to the third angle of the other b, viz. b 26. 1.
the fide AB to the fide CD, and AC to BD, and the angle
BAC equal to the angle BDC: And becaufe the angle ABC is
equal to the angle BCD, and the angle CBD to the angle ACB,
the whole angle ABD is equal to the whole angle ACD: And
the angle BAC has been fhown to be equal to the angle BDC;
therefore the oppofite fides and angles of parallelograms are e-
qual to one another; alfo, their diameter bifects them; for
AB being equal to CD, and BC common, the two AB, BC
are equal to the two DC, CB, each to each; and the angle ABC

C 4   is

c 4. I.

is equal to the angle BCD; therefore the triangle ABC is e-qual ^c to the triangle BCD, and the diameter BC divides the parallelogram ACDB into two equal parts.   Q. E. D.

PROP. XXXV.   THEOR.

See N.

PARALLELOGRAMS upon the fame bafe and between the fame parallels, are equal to one another.

See the 2d and 3d fi-gures.

Let the parallelograms ABCD, EBCF be upon the fame bafe BC, and between the fame parallels AF, BC; the paralle-logram ABCD fhall be equal to the parallelogram EBCF.

If the fides AD, DF of the pa-rallelograms ABCD, DBCF oppofite to the bafe BC be terminated in the fame point D; it is plain that each of the parallelograms is double of the triangle BDC; and they are there-fore equal to one another
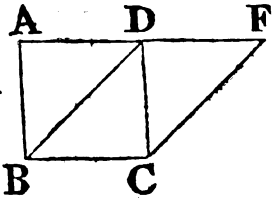
a. 34. I.

But, if the fides AD, EF, oppofite to the bafe BC of the parallelograms ABCD, EBCF, be not terminated in the fame point: then, be-caufe ABCD is a parallelogram, AD is equal ^a to BC, for the fame reafon EF is equal to BC; wherefore AD is equal ^b to EF; and DE is common; therefore the whole, or the remain-der, AE is equal ^c to the whole, or the remainder DF; AB al-fo is equal to DC; and the two EA, AB are therefore equal to

b I. Ax.
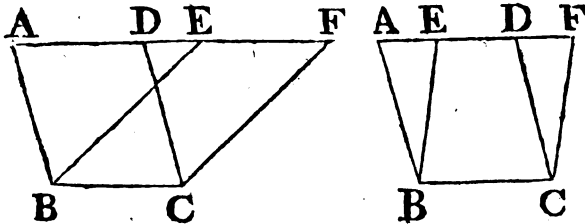
c 2. or 3. Ax.

the two FD, DC, each to each; and the exterior angle FDC is equal ^d to the interior EAB; therefore the bafe EB is equal to the bafe FC, and the triangle EAB equal ^e to the triangle FDC; take the triangle FDC from the trapezium ABCF, and from the fame trapezium take the triangle EAB; the remain-ders therefore are equal ^f, that is, the parallelogram ABCD is equal to the parallelogram EBCF. Therefore parallelograms upon the fame bafe, &c. Q. E. D.

d 29. I.

e 4. I.

f 3. Ax.

                                  P R O P.

PROP. XXXVI.   THEOR.

PARALLELOGRAMS upon equal bafes, and between the fame parallels, are equal to one another.

Let ABCD, EFGH be parallelograms upon equal bafes BC, FG, and between the fame parallels AH, BG ; the parallelogram ABCD is equal to EFGH.
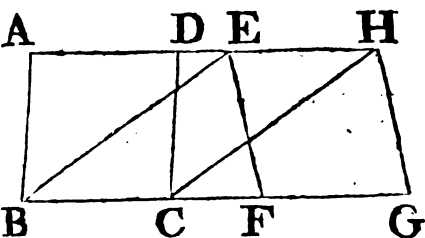
Join BE, CH ; and becaufe BC is equal to FG, and FG to ^a EH, BC is equal to EH ; and they are parallels, and joined towards the fame parts by the ftraight lines BE, CH : But ftraight lines which join equal and parallel ftraight lines towards the fame parts, are themfelves equal and parallel ^b; therefore EB, CH are both equal and parallel, and EBCH is a parallelogram ; and it is equal ^c to ABCD, becaufe it is upon the fame bafe BC, and between the fame parallels BC, AD : For the like reafon, the parallelogram EFGH is equal to the fame EBCH : Therefore alfo the parallelogram ABCD is equal to EFGH.   Wherefore parallelograms, &c. Q. E. D.

a 34. 1.

b 33. 1.

c 35. 1.

PROP. XXXVII.   THEOR.

TRIANGLES upon the fame bafe, and between the fame parallels, are equal to one another.

Let the triangles ABC, DBC be upon the fame bafe BC and between the fame parallels AD, BC : The triangle ABC is equal to the triangle DBC.
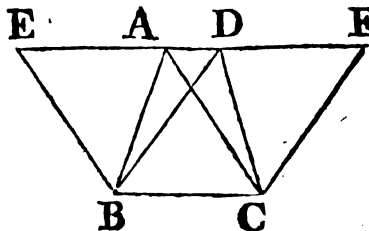
Produce AD both ways to the points E, F, and thro' B draw ^a BE parallel to CA ; and thro' C draw CF parallel to BD : Therefore each of the figures EBCA, DBCF is a parallelogram ; and EBCA is equal ^b to DBCF, becaufe they are upon the fame bafe BC, and between the fame parallels BC, EF ; and the triangle ABC is

a 31. 1.

b 35. 1.

the

the half of the parallelogram EBCA, becaufe the diameter AB
bifects c it ; and the triangle DBC is the half of the parallelo-
gram DBCF, becaufe the diameter DC bifects it : But the
halves of equal things are equal d ; therefore the triangle ABC
is equal to the triangle DBC.   Wherefore triangles, &c.
Q. E. D.

PROP. XXXVIII.   THEOR.

TRIANGLES upon equal bafes, and between the fame
parallels, are equal to one another.

Let the triangles ABC, DEF be upon equal bafes BC, EF,
and between the fame parallels BF, AD : The triangle ABC
is equal to the triangle DEF.

Produce AD both ways to the points G, H, and through B
draw BG parallel a to CA, and through F draw FH parallel to
ED : Then each of
the figures GBCA,
DEFH is a paralle-
logram ; and they
are equal to b one an-
other, becaufe they
are upon equal bafes
BC, EF, and between
the fame parallels

BF, GH ; and the triangle ABC is the half c of the parallelogram
GBCA, becaufe the diameter AB bifects it ; and the triangle
DEF is the half c of the parallelogram DEFH, becaufe the di-
ameter DF bifects it : But the halves of equal things are equal d ;
therefore the triangle ABC is equal to the triangle DEF.
Wherefore triangles, &c· Q. E. D.
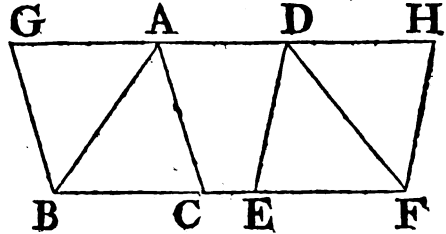
PROP. XXXIX.   THEOR.

EQUAL triangles upon the fame bafe, and upon the
fame fide of it, are between the fame parallels.

Let the equal triangles ABC, DBC be upon the fame bafe
BC, and upon the fame fide of it ; they are between the fame
parallels.

Join AD ; AD is parallel to BC ; for, if it is not, through
the point A draw a AE parallel to BC, and join EC : The tri-
angle

angle. ABC is equal [b] to the triangle EBC, becaufe it is upon the famebafe BC, and between the fame parallels BC, AE : But the triangle ABC is equal to the triangle BDC ; therefore alfo the triangle BDC is e- qual to the triangle EBC, the greater to the lefs, which is impoffible : There- fore AE is not parallel to BC. In the fame manner, it can be demonftrated that no other line but AD is parallel to BC ; AD is therefore parallel to it. Wherefore equal triangles upon, &c. Q. E D.

Book I,
b 37. I,
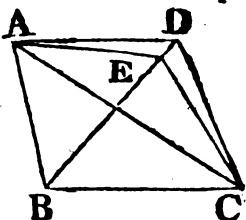
PROP. XL. THEOR.

EQUAL triangles upon equal bafes, in the fame ftraight line, and towards the fame parts, are be- tween the fame parallels.

Let the equal triangles ABC, DEF be upon equal bafes BC, EF, in the fame ftraight line BF, and towards the fame parts ; they are be- tween the fame parallels. Join AD ; AD is paral- lel to BC : For, if it is not, through A draw [a] AG pa- rallel to BF, and join GF : The triangle ABC is equal [b] to the triangle GEF, becaufe they are upon equal bafes BC, EF, and between the fame parallels BF, AG: But the triangle ABC is equal to the triangle DEF ; therefore alfo the triangle DEF is equal to the triangle GEF, the greater to the lefs, which is impoffible : Therefore AG is not parallel to BF : And in the fame manner it can be demonftrated that there is no other paral- lel to it but AD ; AD is therefore parallel to BF. Wherefore equal triangles, &c. Q. E. D.

a 31. I,
b 38. I,
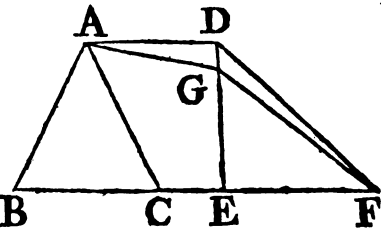
PROP. XLI. THEOR.

IF a parallelogram and triangle be upon the fame bafe, and between the fame parallels ; the parallelogram fhall be double of the triangle.

Let

Let the parallelogram ABCD and the triangle EBC be upon the fame bafe BC, and between the fame parallels BC, AE ; the parallelogram ABCD is double of the triangle EBC.
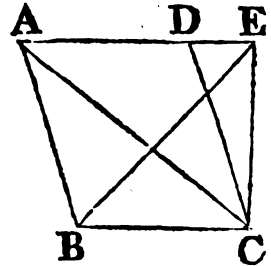
Join AC ; then the triangle ABC
is equal [a] to the triangle EBC, be
caufe they are upon the fame bafe
BC, and between the fame parallels
BC, AE.　But the parallelogram
ABCD is double [b] of the triangle
ABC, becaufe the diameter AC di-
vides it into two equal parts ; where-
fore ABCD is alfo double of the tri-
angle EBC.　Therefore, if a parallelogram, &c. Q. E. D.

a 37. 1.
b 34. 1.

P R O P. XLII. P R O B.

TO defcribe a parallelogram that fhall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle. It is required to defcribe a parallelogram that fhall be equal to the given triangle ABC, and have one of its angles equal to D.

a 10. 1.
b 23. 1.
c 31. 1.

Bifect [a] BC in E, join AE, and at the point E in the ftraight
line EC make [b] the angle CEF equal to D ; and through A draw
[c] AG parallel to EC, and through
C draw CG [c] parallel to EF ;
Therefore FECG is a parallelo-
gram : And becaufe BE is equal
to EC, the triangle ABE is like-
wife equal [d] to the triangle AEC,
fince they are upon equal bafes
BE, EC, and between the fame
parallels BC, AG ; therefore the
triangle ABC is double of the
triangle AEC : And the paral-

d 38. 1.

e 41. 1.
lelogram FECG is likewife double [e] of the triangle AEC, be-
caufe it is upon the fame bafe, and between the fame pa-
rallels : Therefore the parallelogram FECG is equal to the
triangle ABC, and it has one of its angles CEF equal to the
given angle D : Wherefore there has been defcribed a paralle-
logram
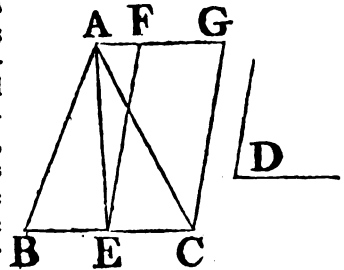
logram FECG equal to a given triangle ABC, having one of
its angles CEF equal to the given angle D.   Which was to be
done.

P R O P   XLIII.   T H E O R.

THE complements of the parallelograms which are
about the diameter of any parallelogram, are equal
to one another.

Let ABCD be a parallelogram, of which the diameter is
AC, and EH, FG the paral-
lelograms about AC, *that is,*
through which AC *paffes,* and
BK, KD the other parallelo
grams which make up the
whole figure ABCD, which
are therefore called the com-
plements: The complement
BK is equal to the comple-
ment KD.
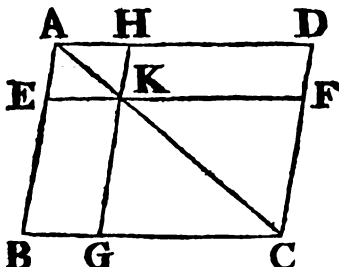
Becaufe ABCD is a paral-
lelogram, and AC its diameter, the triangle ABC is equal [a] to   a 34. 1.
the triangle ADC : And, becaufe EKHA is a parallelogram, the
diameter of which is AK, the triangle AEK is equal to the tri-
angle AHK : By the fame reafon, the triangle KGC is equal to
the triangle KFC : Then, becaufe the triangle AEK is equal to
the triangle AHK, and the triangle KGC to KFC ; the triangle
AEK, together with the triangle KGC is equal to the triangle
AHK together with the triangle KFC : But the whole triangle
ABC is equal to the whole ADC ; therefore the remaining
complement BK is equal to the remaining complement KD.
Wherefore the complements, &c.   Q. E. D.

P R O P.   XLIV.   P R O B.

TO a given ftraight line to apply a parallelogram,
which fhall be equal to a given triangle, and have
one of its angles equal to a given rectilineal angle.

Let AB be the given ftraight line, and C the given triangle,
and D the given rectilineal angle.   It is required to apply to
the ftraight line AB a parallelogram equal to the triangle C,
and having an angle equal to D.

                                             Make

Make ᵃ the parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D, so that BE be in the same straight line with AB,
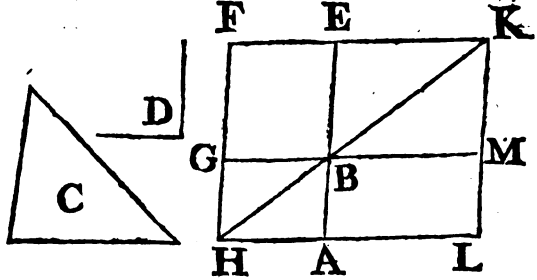
b 31. I.

and produce FG to H ; and thro' A draw ᵇ AH parallel to BG or EF, and join HB.   Then, becaufe the ftraight line HF falls upon the parallels AH, EF, the angles AHF, HFE, are toge-

c 29. I.

ther equal ᶜ to two right angles ; wherefore the angles BHF, HFE are leffer than two right angles : But ftraight lines which with another ftraight line make the interior angles upon the

d 12. Ax.

fame fide lefs than two right angles, do meet ᵈ if produced far enough :   Therefore HB, FE fhall meet, if produced ; let them meet in K, and through K draw KL parallel to EA or FH, and produce HA, GB to the points L, M :   Then HLKF is a paral-lelogram, of which the diameter is HK, and AG, ME are the parallelograms about HK ; and LB, BF are the complements ;

e 43. I.

therefore LB is equal ᵉ to BF: But BF is equal to the triangle C ; wherefore LB is equal to the triangle C : And becaufe the angle

f 15. I.

GBE is equal ᶠ to the angle ABM, and likewife to the angle D ; the angle ABM is equal to the angle D : Therefore the paralle-logram LB is applied to the ftraight line AB, is equal to the triangle C, and has the angle ABM equal to the angle D : Which was to be done.

P R O P. XLV.   P R O B.

TO defcribe a parallelogram equal to a given rectili-neal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle. It is required to defcribe a parallelogram e-qual to ABCD, and having an angle equal to E.

a 42. I.

Join DB, and defcribe ᵃ the parallelogram FH equal to the tri-angle ADB, and having the angle HKF equal to the angle E ;

b 44. I.

and to the ftraight line GH apply ᵇ the parallelogram GM equal

to

to the triangle DBC, having the angle GHM equal to the angle
E; and becaufe the angle E is equal to each of the angles
FKH, GHM, the angle FKH is equal to GHM; add to
each of thefe the angle KHG; therefore the angles FKH,
KHG are equal
to the angles
KHG GHM;
but FKH, KHG
are equal ᶜ to two
right angles;
therefore alfo
KHG, GHM
are equal to two
right angles; and
becaufe at the
point H in the ftraight line GH, the two ftraight lines KH, HM,
upon the oppofite fides of it make the adjacent angles equal to
two right angles, KH is in the fame ftraight line ᵈ with HM; d 14. 1.
and becaufe the ftraight line HG meets the parallels KM, FG,
the alternate angles MHG, HGF are equal ᶜ: Add to each of
thefe, the angle HGL: Therefore the angles MHG, HGL are
equal to the angles HGF, HGL: But the angles MHG, HGL
are equal ᶜ to two right angles; wherefore alfo the angles HGF,
HGL are equal to two right angles, and FG is therefore in the
fame ftraight line with GL: And becaufe KF is parallel to
HG, and HG to ML; KF is parallel ᶜ to ML: And KM, FL e 30. 1.
are parallels; wherefore KFLM is a parallelogram; and becaufe
the triangle ABD is equal to the parallelogram HF, and the
triangle DBC to the parallelogram GM; the whole rectili-
neal figure ABCD is equal to the whole parallelogram KFLM;
therefore the parallelogram KFLM has been defcribed equal to
the given rectilineal figure ABCD, having the angle FKM e-
qual to the given angle E. Which was to be done.
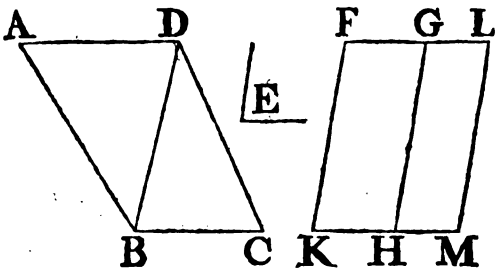
COR. From this it is manifeft how to a given ftraight line to
apply a parallelogram, which fhall have an angle equal to a
given rectilineal angle, and fhall be equal to a given rectilineal
figure, viz. by applying ᵇ to the given ftraight line, a parallelo- b 44. 1.
gram equal to the firft triangle ABD, and having an angle e-
qual to the given angle.

<p style="text-align: right">PROP.</p>

c 29. 1.

PROP. XLVI. PROB.

To defcribe a fquare upon a given ftraight line.

Let AB be the given ftraight line ; it is required to defcribe a fquare upon AB.

From the point A draw a AC at right angles to AB; and make b AD equal to AB, and through the point D draw DE parallel c to AB, and through B draw BE parallel to AD ; therefore ADEB is a parallelogram ; whence AB is equal d to DE, and AD to BE: But BA is equal to AD ; therefore the four ftraight lines BA, AD, DE, EB are equal to one another, and the parallelogram ADEB is equilateral, likewife all its angles are right angles ; becaufe the ftraight line AD meeting the parallels AB, DE, the angles BAD, ADE are e- qual e to two right angles ; but BAD is a right angle ; therefore alfo ADE is a right angle ; but the oppofite angles of parallelograms are equal d ; therefore each of the oppofite angles ABE, BED is a right angle ; wherefore the figure ADEB is rectangular, and it has been demonftrated that it is equilateral ; it is therefore a fquare, and it is defcribed upon the given ftraight line AB : Which was to be done.

a II. I.
b 3. I.
c 31. I.
d 34. I.

e 29. I.

COR. Hence every parallelogram that has one right angle has all its angles right angles.

PROP. XLVII. THEOR.

IN any right angled triangle, the fquare which is de- fcribed upon the fide fubtending the right angle, is equal to the fquares defcribed upon the fides which con- tain the right angle.

Let ABC be a right angled triangle having the right angle BAC ; the fquare defcribed upon the fide BC is equal to the fquares defcribed upon BA, AC.

a 46. I.

On BC defcribe a the fquare BDEC, and on BA, AC the fquares
GB,

GB, HC; and through A draw[b] AL parallel to BD or CE, and join AD, FC; then, becaufe each of the angles BAC, BAG is a right angle[c], the two ftraight lines AC, AG up-on the oppofite fides of AB, make with it at the point A the adjacent angles equal to two right angles; therefore CA is in the fame ftraight line[d] with AG; for the fame reafon, AB and AH are in the fame ftraight line; and becaufe the angle DBC is e-qual to the angle FBA, each of them being a right angle, add to each the angle ABC, and the whole angle DBA is

equal[e] to the whole FBC; and becaufe the two fides AB, BD are equal to the two FB, BC, each to each; and the angle DBA equal to the angle FBC; therefore the bafe AD is e-qual[f] to the bafe FC, and the triangle ABD to the triangle FBC: Now the parallelogram BL is double[g] of the triangle ABD, becaufe they are upon the fame bafe BD, and between the fame parallels, BD, AL; and the fquare GB is double of the triangle FBC, becaufe thefe alfo are upon the fame bafe FB, and between the fame parallels FB, GC. But the doubles of equals are equal[h] to one another: Therefore the parallelo-gram BL is equal to the fquare GB: And in the fame manner, by joining AE, BK, it is demonftrated that the parallelogram CL is equal to the fquare HC: Therefore the whole fquare BDEC is equal to the two fquares GB, HC; and the fquare BDEC is defcribed upon the ftraight line BC, and the fquares GB, HC upon BA, AC: Wherefore the fquare upon the fide BC is equal to the fquares upon the fides BA, AC. Therefore, in any right angled triangle, &c. Q. E. D.
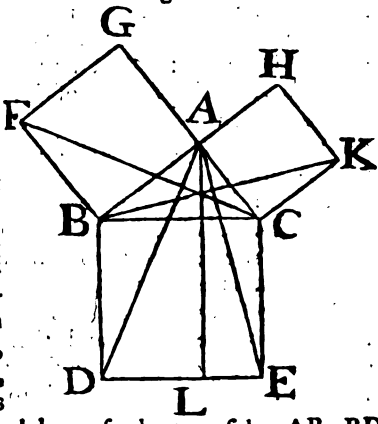
PROP. XLVIII. THEOR.

IF the fquare defcribed upon one of the fides of a tri-angle, be equal to the fquares defcribed upon the o-ther two fides of it; the angle contained by thefe two fides is a right angle.

If the fquare defcribed upon BC, one of the fides of the tri-
angle ABC, be equal to the fquares upon the other fides BA,
AC; the angle BAC is a right angle.

a II. I.
From the point A draw ª AD at right angles to AC, and
make AD equal to BA, and join DC: Then, becaufe DA is
equal to AB, the fquare of DA is equal
to the fquare of AB: To each of thefe
add the fquare of AC; therefore the fquares
of DA, AC, are equal to the fquares of
BA, AC: But the fquare of DC is equal

b 47. I.
ᵇ to the fquares of DA, AC, becaufe DAC
is a right angle; and the fquare of BC, by
hypothefis, is equal to the fquares of BA,
AC; therefore the fquare of DC is equal
to the fquare of BC; and therefore alfo
the fide DC is equal to the fide BC. And
becaufe the fide DA is equal to AB, and AC common to the
two triangles DAC, BAC, the two DA, AC are equal to the
two BA, AC; and the bafe DC is equal to the bafe BC; there-

c 8. I.
fore the angle DAC is equal ᶜ to the angle BAC; But DAC
is a right angle; therefore alfo BAC is a right angle. There-
fore, if the fquare, &c. Q. E. D.

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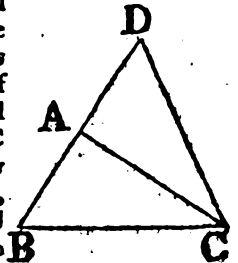
ELEMENTS

OF

EUCLID.

BOOK II.

DEFINITIONS.

I.

EVERY right angled parallelogram is said to be contained by any two of the ftraight lines which contain one of the right angles.

II.

In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon. ‘ Thus the parallelogram HG, together with the complements AF, FC, is the gnomon, which is more briefly expreffed by the letters AGK, or EHC which are at the oppofite angles of the parallelograms which make the gnomon.’
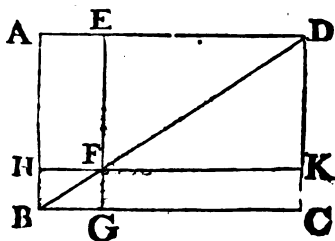
PROP. I. THEOR.

IF there be two ftraight lines, one of which is divided into any number of parts ; the rectangle contained by the two ftraight lines, is equal to the rectangles contained by the undivided line, and the feveral parts of the divided line.

D 2 Let

Let A and BC be two ſtraight lines ; and let BC be divided into any parts in the points D, E ; the rectangle contained by the ſtraight lines A, BC is equal to the rectangle contained by A, BD, together with that contained by A, DE, and that contained by A, EC.

a 11. 1.

From the point B draw ᵃ BF at right angles to BC, and make BG equal ᵇ to A ; and through G draw ᶜ GH parallel to BC ; and through D, E, C draw ᶜ DK EL, CH parallel to BG ; then the

b 3. 1.

c 31. 1.

rectangle BH is equal to the rectangles BK, DL, EH ; and BH is contained by A, BC, for it is contained by GB, BC, and GB is equal to A ; and BK is contained by A, BD, for it is contained by GB, BD, of which GB is equal to A ; and DL is contained by A, DE, becauſe DK, that is, ᵈ BG, is equal to A ; and in like manner the rectangle EH is contained by A, EC : Therefore the rectangle contained by A, BC is equal to the ſeveral rectangles contained by A, BD, and by A, DE ; and alſo by A, EC. Wherefore, if there be two ſtraight lines, &c. Q. E. D.

d 34. 1.
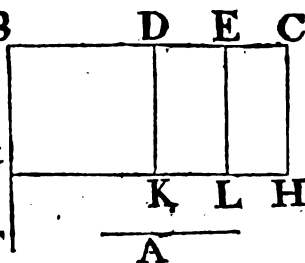
PROP. II. THEOR.

IF a ſtraight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the ſquare of the whole line.

Let the ſtraight line AB be divided into any two parts in the point C ; the rectangle contained by AB, BC, together with the rectangle * AB, AC, ſhall be equal to the ſquare of AB.

a 46. 1.

b 31. 1.

Upon AB deſcribe ᵃ the ſquare ADEB, and through C draw ᵇ CF, parallel to AD, or BE ; then AE is equal to the rectangles AF, CE ; and AE is the ſquare of AB ;
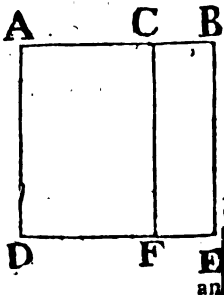
* N. B. To avoid repeating the word *contained* too frequently, the rectangle contained by two ſtraight lines AB, AC is ſometimes ſimply called the rectangle AB, AC.

and AF is the rectangle contained by BA, AC; for it is contained by DA, AC, of which AD is equal to AB; and CE is contained by AB, BC, for BE is equal to AB; therefore the rectangle contained by AB, AC, together with the rectangle AB, BC, is equal to the square of AB. If therefore a straight line, &c. Q. E. D.

PROP. III.   THEOR.

IF a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square of the foresaid part.

Let the straight line AB be divided into any two parts in the point C; the rectangle AB, BC is equal to the rectangle AC, CB, together with the square of BC.

Upon BC describe a the square CDEB, and produce ED to F, and through A draw b AF parallel to CD or BE; then the rectangle AE is equal to the rectangles AD, CE; and AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC; and AD is contained by AC, CB, for CD is equal to CB; and DB is the square of BC; therefore the rectangle AB, BC is equal to the rectangle AC, CB together with the square of BC. If therefore a straight line, &c. Q. E. D.

a 46. 1.

b 31. 1.
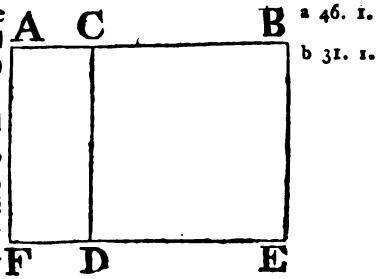
PROP. IV.   THEOR.

IF a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let the straight line AB be divided into any two parts in C; the square of AB is equal to the squares of AC, CB and to twice the rectangle contained by AC, CB.

                                              Upon

a. 46. 1.
b 31. 1.
c 29. 1.
d 5. 1.

Upon AB defcribes the fquare ADEB, and join BD, and through C draw b CGF parallel to AD or BE, and through G draw HK parallel to AB or DE: And becaufe CF is parallel to AD, and BD falls upon them, the exterior angle BGC is equal c to the interior and oppofite angle ADB; but ADB is equal d to the angle ABD, becaufe BA is equal to AD, being fides of a

fquare; wherefore the angle CGB is equal to the angle GBC; and there-

e 6. 1.

fore the fide BC is equal e to the

f 34. 1.

fide CG: But CB is equal f alfo to GK, and CG to BK; wherefore the figure CGKB is equilateral: It is likewife rectangular; for CG is parallel to BK, and CB meets them; the angles KBC, GCB are therefore equal to two right angles; and KBC

is a right angle; wherefore GCB is a right angle; and therefore alfo the angles f CGK, GKB oppofite to thefe are right angles, and CGKB is rectangular: But it is alfo equilateral, as was demonftrated; wherefore it is a fquare, and it is upon the fide CB: For the fame reafon HF alfo is a fquare, and it is upon the fide HG which is equal to AC: Therefore HF, CK are the fquares of AC, CB; and becaufe the complement AG is equal

g 43. 1.

g to the complement GE, and that AG is the rectangle contained by AC, CB, for GC is equal to CB; therefore GE is alfo equal to the rectangle AC, CB; wherefore AG, GE are equal to twice the rectangle AC, CB: And HF, CK are the fquares of AC, CB; wherefore the four figures HF, CK, AG, GE are equal to the fquares of AC, CB, and to twice the rectangle AC, CB: But HF, CK, AG, GE make up the whole figure ADEB, which is the fquare of AB: Therefore the fquare of AB is equal to the fquares of AC, CB and twice the rectangle AC, CB. Wherefore, if a ftraight line, &c. Q. E. D.
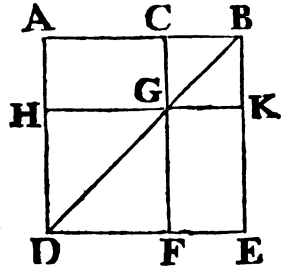
Cor. From the demonftration, it is manifeft, that the parallelograms about the diameter of a fquare are likewife fquares.

PROP. V. THEOR.

IF a ftraight line be divided into two equal parts, and alfo into two unequal parts; the rectangle contained by the unequal parts, together with the fquare of the line between the points of fection, is equal to the fquare of half the line.

Let the ftraight line AB be divided into two equal parts in the point C, and into two unequal parts at the point D; the rectangle AD, DB, together with the fquare of CD, is equal to the fquare of CB.

Upon CB defcribe ᵃ the fquare CEFB, join BE, and through ᵃ 46. 1.
D draw ᵇ DHG parallel to CE or BF; and through H draw ᵇ 31. 1.
KLM parallel to CB or EF; and alfo through A draw AK parallel to CL or BM: And becaufe the complement CH is e- ᶜ 43. 1.
qual ᶜ to the complement HF, to each of thefe add DM;
therefore the whole CM
is equal to the whole DF;
but CM is equal ᵈ to AL, A   C   D B ᵈ 36. 1.
becaufe AC is equal to
CB; therefore alfo AL is
equal to DF. To each of
thefe add CH, and the
whole AH is equal to
DF and CH: But AH is
the rectangle contained by
AD, DB, for DH is equal
ᵉ to DB; and DF together with CH is the gnomon CMG; ᵉ Cor. 4.
therefore the gnomon CMG is equal to the rectangle AD, DB: 2.
To each of thefe add LG, which is equal ᵉ to the fquare of CD;
therefore the gnomon CMG, together with LG, is equal to the
rectangle AD, DB, together with the fquare of CD: But the
gnomon CMG and LG make up the whole figure CEFB, which
is the fquare of CB: Therefore the rectangle AD, DB, together
with the fquare of CD, is equal to the fquare of CB. Wherefore,
if a ftraight line, &c. Q. E. D.
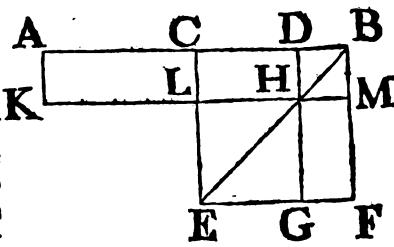
From this propofition it is manifeft, that the difference of the
fquares of two unequal lines AC, CD, is equal to the rectangle
contained by their fum and difference.

D   PROP.

PROP. VI. THEOR.

IF a ftraight line be bifected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the fquare of half of the line bifected, is equal to the fquare of the ftraight line which is made up of the half and the part produced.

Let the ftraight line AB be bifected in C, and produced to the point D; the rectangle AD, DB, together with the fquare of CB, is equal to the fquare of CD.

a 46. 1.
b 31. 1.

Upon CD defcribe ª the fquare CEFD, join DE, and through B draw ᵇ BHG parallel to CE or DF, and through H draw KLM parallel to AD or EF, and alfo through A draw AK parallel to CL or DM: And becaufe AC is equal to CB the rectangle AL is equal ᶜ to CH; but CH is equal ᵈ to HF; there fore alfo AL is equal to HF: To each of thefe add CM; therefore the whole AM is equal to the gnomon CMG: And AM is the rectangle contained by AD, DB, for DM is equal to DB: Therefore the gnomon CMG is equal to the rect-angle AD, DB: Add to each of thefe LG, which is equal to the fquare of CB; therefore the rectangle AD, DB, together with the fquare of CB, is equal to the gnomon CMG and the figure LG: But the gnomon CMG and LG make up the whole figure CEFD, which is the fquare of CD; therefore the rectangle AD, DB, together with the fquare of CB, is e-qual to the fquare of CD. Wherefore, if a ftraight line, &c. Q. E. D.
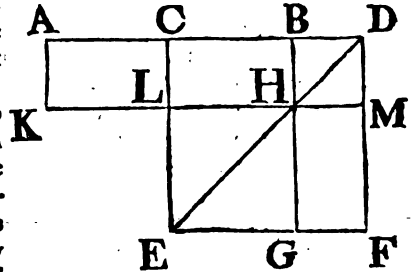
c 36. 1.
d 43. 1.

e Cor. 4. 2.

PROP. VII. THEOR.

IF a ftraight line be divided into any two parts, the fquares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the fquare of the other part.

Let the ftraight line AB be divided into any two parts in the

the point C; the fquares of AB, BC are equal to twice the
rectangle AB, BC together with the fquare of AC.

Upon AB defcribe ᵃ the fquare ADEB, and conftruct the ᵃ 46. 1.
figure as in the preceding propofitions : And becaufe AG is
equal ᵇ to GE, add to each of them CK; the whole AK is ᵇ 43. 1.
therefore equal to the whole CE;
therefore AK, CE are double of
AK : But AK, CE are the gnomon
AKF together with the fquare CK;
therefore the gnomon AKF, toge-
ther with the fquare CK, is double
of AK : But twice the rectangle AB,
BC is double of AK, for BK is e-
qual ᶜ to BC: Therefore the gno-
mon AKF, together with the fquare
CK, is equal to twice the rectangle
AB, BC: To each of thefe equals
add HF, which is equal to the fquare of AC; therefore the
gnomon AKF, together with the fquares CK, HF, is equal to
twice the rectangle AB, BC and the fquare of AC: But the
gnomon AKF, together with the fquares CK, HF, make up the
whole figure ADEB and CK, which are the fquares of AB
and BC: Therefore the fquares of AB and BC are equal to
twice the rectangle AB, BC, together with the fquare of AC.
Wherefore, if a ftraight line, &c. Q. E. D.
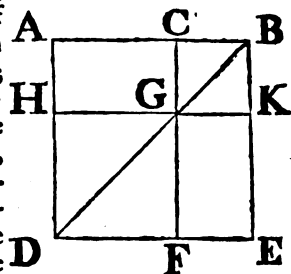
c Cor. 4. 2.

PROP. VIII. THEOR.

IF a ftraight line be divided into any two parts, four
times the rectangle contained by the whole line, and
one of the parts, together with the fquare of the other
part, is equal to the fquare of the ftraight line which is
made up of the whole and that part.

Let the ftraight line AB be divided into any two parts in the
point C; four times the rectangle AB, BC, together with the
fquare of AC, is equal to the fquare of the ftraight line made up
of AB and BC together.

Produce AB to D, fo that BD be equal to CB, and upon
AD defcribe the fquare AEFD; and conftruct two figures
fuch as in the preceding. Becaufe CB is equal to BD, and
that CB is equal ᵃ to GK, and BD to KN; therefore GK is ᵃ 34. 1.
equal

equal to KN: For the same reason, PR is equal to RO; and because CB is equal to BD, and GK to KN, the rectangle CK is equal [b] to BN, and GR to RN: But CK is equal [c] to RN, because they are the complements of the parallelogram CO; therefore also BN is equal to GR; and the four rectangles BN, CK, GR, RN are therefore equal to one another, and so are quadruple of one of them CK: Again, because CB is equal to BD, and that BD is

equal [d] to BK, that is, to CG;

and CB equal to GK, that [d] is, to GP; therefore CG is equal to GP: And because CG is equal to GP, and PR to RO, the rectangle AG is equal to MP, and PL to

RF: But MP is equal [e] to PL, because they are the complements of the parallelogram ML; wherefore AG is equal also to RF: Therefore the four rectangles AG, MP, PL, RF are equal to one another, and so are quadruple of one of them AG. And it was demonstrated, that the four CK, BN, GR, RN are quadruple of CK: Therefore the eight rectangles which contain the gnomon AOH, are quadruple of AK: And because AK is the rectangle contained by AB, BC, for BK is equal to BC, four times the rectangle AB, BC is quadruple of AK: But the gnomon AOH was demonstrated to be quadruple of AK; therefore four times the rectangle AB, BC is equal to the gnomon AOH.

To each of these add XH, which is equal [d] to the square of AC: Therefore four times the rectangle AB, BC, together with the square of AC, is equal to the gnomon AOH and the square XH: But the gnomon AOH and XH make up the figure AEFD which is the square of AD: Therefore four times the rectangle AB, BC, together with the square of AC, is equ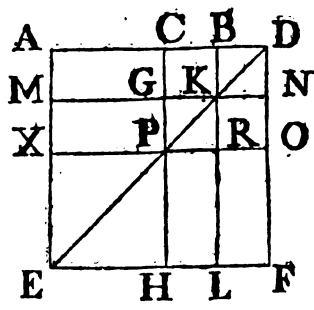al to the square of AD, that is, of AB and BC added together in one straight line. Wherefore, if a straight line, &c. Q. E. D.

PROP.

PROP. IX. THEOR.

IF a ftraight line be divided into two equal, and alfo in-
to two unequal parts; the fquares of the two unequal
parts are together double of the fquare of half the
line, and of the fquare of the line between the points of
fection.

Let the ftraight line AB be divided at the point C into two
equal, and at D into two unequal parts : The fquares of AD,
DB are together double of the fquares of AC, CD.

From the point C draw ᵃ CE at right angles to AB, and a 11. 1.
make it equal to AC or CB, and join EA, EB; through D draw
ᵇ DF parallel to CE, and through F draw FG parallel to AB; b 31. 1.
and join AF: Then, becaufe AC is equal to CE, the angle
EAC is equal ᶜ to the angle AEC; and becaufe the angle c 5. 1.
ACE is a right angle, the two others AEC, EAC together
make one right angle ᵈ; and they are equal to one another; d 32. 1.
each of them therefore is half
of a right angle. For the fame
reafon each of the angles CEB,
EBC is half a right angle; and
therefore the whole AEB is a
right angle: And becaufe the angle
GEF is half a right angle, and
EGF a right angle, for it is e-
qual ᵉ to the interior and oppo-
fite angle ECB, the remaining angle EFG is half a right angle;
therefore the angle GEF is equal to the angle EFG, and the
fide EG equal ᶠ to the fide GF: Again, becaufe the angle at B f 6. 1.
is half a right angle, and FDB a right angle, for it is equal
ᵉ to the interior and oppofite angle ECB, the remaining angle
BFD is half a right angle; therefore the angle at B is equal
to the angle BFD, and the fide DF to ᶠ the fide DB: And be-
caufe AC is equal to CE, the fquare of AC is equal to the
fquare of CE; therefore the fquares of AC, CE are double of
the fquare of AC: But the fquare of EA is equal ᵍ to the g 47. 1.
fquares of AC, CE, becaufe ACE is a right angle; therefore
the fquare of EA is double of the fquare of AC: Again, be-
caufe EG is equal to GF, the fquare of EG is equal to the
fquare of GF; therefore the fquares of EG, GF are double of

the
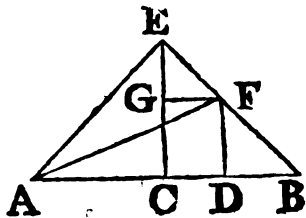
e 29. 1.

the fquare of GF; but the fquare of EF is equal to the fquares
of EG, GF; therefore the fquare of EF is double of the fquare
GF: And GF is equal ʰ to CD; therefore the fquare of EF is
double of the fquare of CD: But the fquare of AE is likewife
double of the fquare of AC; therefore the fquares of AE, EF
are double of the fquares of AC, CD: And the fquare of AF is
equal ⁱ to the fquares of AE, EF, becaufe AEF is a right angle;
therefore the fquare of AF is double of the fquares of AC,
CD: But the fquares of AD, DF are equal to the fquare of
AF, becaufe the angle ADF is a right angle; therefore the
fquares of AD, DF are double of the fquares of AC, CD: And
DF is equal to DB; therefore the fquares of AD, DB are
double of the fquares of AC, CD. If therefore a ftraight line,
&c. Q. E. D.

PROP. X. THEOR.

IF a ftraight line be bifected, and produced to any point,
the fquare of the whole line thus produced, and the
fquare of the part of it produced, are together double
of the fquare of half the line bifected, and of the
fquare of the line made up of the half and the part pro-
duced.

·Let the ftraight line AB be bifected in C, and produced to
the point D; the fquares of AD, DB are double of the fquares
of AC, CD.
From the point C draw ᵃ CE at right angles to AB: And
make it equal to AC or CB, and join AE, EB; through E draw
ᵇ EF parallel to AB, and through D draw DF parallel to CE:
And becaufe the ftraight line EF meets the parallels EC, FD, the
angles CEF, EFD are equal ᶜ to two right angles; and therefore
the angles BEF, EFD are lefs than two right angles: But ftraight
lines which with another ftraight line make the interior angles
upon the fame fide lefs than two right angles, do meet ᵈ if pro-
duced far enough: Therefore EB, FD fhall meet, if produced,
toward B, D: Let them meet in G, and join AG: Then, becaufe
AC is equal to CE, the angle CEA is equal ᵉ to the angle
EAC; and the angle ACE is a right angle; therefore each of the
angles CEA, EAC is half a right angle ᶠ: For the fame reafon,
each

each of the angles CEB, EBC is half a right angle; therefore
AEB is a right angle: And becaufe EBC is half a right angle,
DBG is alfo f half a right angle, for they are vertically oppô- f 15. 1.
fite; but BDG is a right angle, becaufe it is equal c to the al- c 29. 1.
ternate angle DCE; therefore the remaining angle DGB is
half a right angle, and is therefore equal to the angle DBG;
wherefore alfo the fide BD is equal g to the fide DG: Again, g 6. 1.

becaufe EGF is half a
right angle, and that
the angle at F is a right
angle, becaufe it is e-
qual h to the oppofite   h 34. 1.
angle ECD, the remain-
ing angle FEG is half a
right angle, and equal
to the angle EGF,
wherefore alfo the fide
GF is equal g to the fide FE. And becaufe EC is equal to
CA, the fquare of EC is equal to the fquare of CA; therefore
the fquares of EC, CA are double of the fquare of CA: But
the fquare of EA is equal i to the fquares of EC, CA; there- i 47. 1.
fore the fquare of EA is double of the fquare of AC: Again,
becaufe GF is equal to FE, the fquare of GF is equal to the
fquare of FE; and therefore the fquares of GF, FE are dou-
ble of the fquare of EF: But the fquare of EG is equal i to
the fquares of GF, FE; therefore the fquare of EG is double
of the fquare of EF: And EF is equal to CD; wherefore the
fquare of EG is double of the fquare of CD: But it was demon-
ftrated, that the fquare of EA is double of the fquare of AC;
therefore the fquares of AE, EG are double of the fquares of
AC, CD: And the fquare of AG is equal i to the fquares of
AE, EG; therefore the fquare of AG is double of the fquares of
AC, CD: But the fquares of AD, DG are equal i to the
fquare of AG; therefore the fquares of AD, DG are double
of the fquares of AC, CD: But DG is equal to DB; therefore
the fquares of AD, DB are double of the fquares of AC, CD:
Wherefore, if a ftraight line, &c. Q E D.
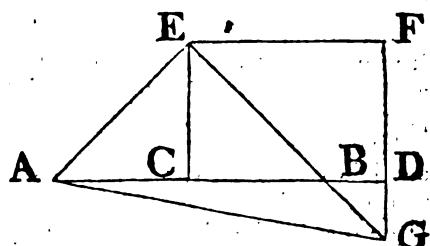
PROP. XI. PROB.

TO divide a given ſtraight line into two parts, ſo that
the rectangle contained by the whole, and one of
the parts, ſhall be equal to the ſquare of the other
part.

Let AB be the given ſtraight line ; it is required to divide
it into two parts, ſo that the rectangle contained by the whole,
and one of the parts, ſhall be equal to the ſquare of the other
part.

a 46. 1.   Upon AB deſcribe a the ſquare ABDC ; biſect b AC in E,
b 10. 1.   and join BE ; produce CA to F, and make c EF equal to EB ;
c 3. 1.   and upon AF deſcribe a the ſquare FGHA ; AB is divided in
H ſo, that the rectangle AB, BH is equal to the ſquare of AH.
     Produce GH to K : Becauſe the ſtraight line AC is biſected
in E, and produced to the point F, the rectangle CF, FA, to-
d 6. 2.   gether with the ſquare of AE, is equal d to the ſquare of EF :
But EF is equal to EB ; therefore the rectangle CF, FA, toge-
ther with the ſquare of AE, is equal to the ſquare of EB : And the
e 47. 1.   ſquares of BA, AE are equal e to the
ſquare of EB, becauſe the angle EAB
is a right angle ; therefore the rect-
angle CF, FA, together with the ſquare
of AE, is equal to the ſquares of BA,
AE : Take away the ſquare of AE,
which is common to both, therefore
the remaining rectangle CF, FA is e-
qual to the ſquare of AB : And the fi-
gure FK is the rectangle contained by
CF, FA, for AF is equal to FG ; and
AD is the ſquare of AB ; therefore
FK is equal to AD : Take away the
common part AK, and the remainder
FH is equal to the remainder HD :
And HD is the rectangle contained by AB, BH, for AB is e-
qual to BD ; and FH is the ſquare of AH : Therefore the
rectangle AB, BH is equal to the ſquare of AH : Wherefore
the ſtraight line AB is divided in H ſo, that the rectangle AB,
BH is equal to the ſquare of AH.   Which was to be done.
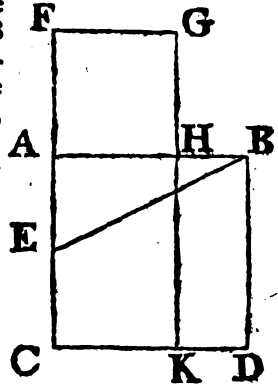
PROP.

PROP. XII. THEOR.

IN obtuse angled triangles, if a perpendicular be drawn from any of the acute angles to the oppofite fide produced, the fquare of the fide fubtending the obtufe angle is greater than the fquares of the fides containing the obtufe angle, by twice the rectangle contained by the fide upon which, when produced, the perpendicular falls, and the ftraight line intercepted without the triangle between the perpendicular and the obtufe angle.

Let ABC be an obtufe angled triangle, having the obtufe angle ACB, and from the point A let AD be drawn [a] perpendicular to BC produced: The fquare of AB is greater than the fquares of AC, CB by twice the rectangle BC, CD.

Becaufe the ftraight line BD is divided into two parts in the point C, the fquare of BD is equal [b] to the fquares of BC, CD, and twice the rectangle BC, CD: To each of thefe equals add the fquare of DA; and the fquares of BD, DA are equal to the fquares of BC, CD, DA, and twice the rectangle BC, CD: But the fquare of BA is equal [c] to the fquares of BD, DA, becaufe the angle at D is a right angle; and the fquare of CA is equal [c] to the fquares of CD, DA: Therefore the fquare of BA is equal to the fquares of BC, CA, and twice the rectangle BC, CD; that is, the fquare of BA is greater than the fquares of BC, CA, by twice the rectangle BC, CD. Therefore, in obtufe angled triangles, &c. Q. E. D.

a 12. 1.

b 4. 2.

c 47. 1.

PROP.

PROP. XIII. THEOR.

See N.

IN every triangle, the square of the side subtending any of the acute angles, is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle, and the acute angle.

a 12. 1.

Let ABC be any triangle, and the angle at B one of its acute angles, and upon BC, one of the sides containing it, let fall the perpendicular a AD from the opposite angle: The square of AC, opposite to the angle B, is less than the squares of CB, BA by twice the rectangle CB, BD.

First, Let AD fall within the triangle ABC; and because the straight line CB is divided into two parts in the point D, the

b 7. 2.

squares of CB, BD are equal b to twice the rectangle contained by CB, BD, and the square of DC: To each of these equals add the square of AD; therefore the squares of CB, BD, DA are equal to twice the rectangle CB, BD, and the squares of AD, DC: But the square of AB is equal

c 47. 1.
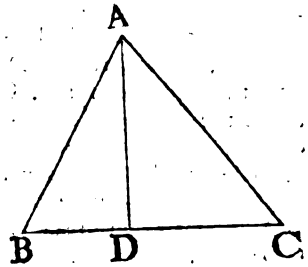
c to the squares of BD, DA, because the angle BDA is a right angle; and the square of AC is equal to the squares of AD, DC: Therefore the squares of CB, BA are equal to the square of AC, and twice the rectangle CB, BD; that is, the square of AC alone is less than the squares of CB, BA by twice the rectangle CB, BD.

Secondly, Let AD fall without the triangle ABC: Then, be-

d 16. 1.

cause the angle at D is a right angle, the angle ACB is greater d than a right angle; and there-

e 12. 2.

fore the square of AB is equal e to the squares of AC, CB, and twice the rectangle BC, CD: To these equals add the square of BC, and the
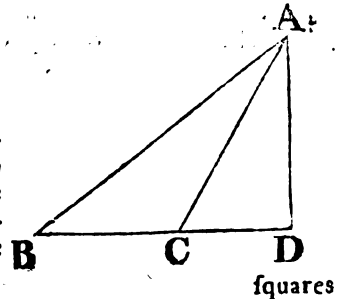
squares

fquares of AB, BC are equal to the fquare of AC, and twice
the fquare of BC, and twice the rectangle BC, CD: But be-
caufe BD is divided into two parts in C, the rectangle DB, BC
is equal f to the rectangle BC, CD and the fquare of BC: And f 3. 2.
the doubles of thefe are equal: Therefore the fquares of AB,
BC are equal to the fquare of AC, and twice the rectangle
DB, BC: Therefore the fquare of AC alone is lefs than the
fquares of AB, BC by twice the rectangle DB, BC.

Laftly, Let the fide AC be perpendicular to
BC; then is BC the ftraight line between the
perpendicular and the acute angle at B; and it
is manifeft that the fquares of AB, BC are e-
qual c to the fquare of AC, and twice the
fquare of BC: Therefore, in every triangle, &c.
Q. E. D.

c 47. 1.

P R O P. XIV.   P R O B.

TO defcribe a fquare that fhall be equal to a given
rectilineal figure.

Let A be the given rectilineal figure; it is required to de-
fcribe a fquare that fhall be equal to A.
Defcribe a the rectangular parallelogram BCDE equal to the a 45. 1.
rectilineal figure A. If then the fides of it BE, ED are equal
to one another, it
is a fquare, and
what was required
is now done: But
if they are not e-
qual, produce one
of them BE to F,
and make EF equal
to ED, and bifect
BF in G; and from
the centre G, at the diftance GB, or GF, defcribe the femi-
circle BHF, and produce DE to H, and join GH: Therefore,
becaufe the ftraight line BF is divided into two equal parts in
the point G, and into two unequal at E, the rectangle BE,
EF, together with the fquare of EG, is equal b to the fquare of b 5. 2.
GF: But GF is equal to GH; therefore the rectangle BE, EF,
E   to-

together with the square of EG, is equal to the square of GH :
But the squares of HE, EG are equal c to the square of GH :
Therefore the rectangle BE, EF, together with the square of
EG, is equal to the squares of HE, EG : Take away the square
of EG, which is common to both ; and the remaining rect-
angle BE, EF is equal to the square of EH : But the rectangle
contained by BE, EF is the parallelogram BD, because EF is
equal to ED ; therefore BD is equal to the square of EH ; but BD
is equal to the rectilineal figure A ; therefore the rectilineal fi-
gure A is equal to the square of EH : Wherefore a square has
been made equal to the given rectilineal figure A, viz. the square
described upon EH. Which was to be done.
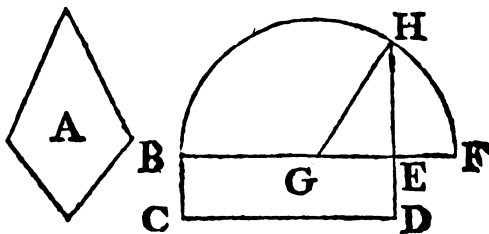
THE

T H E

E L E M E N T S

O F

E U C L I D.

B O O K   III.

D E F I N I T I O N S.

I.

EQUAL circles are thofe of which the diameters are equal,
or from the centres of which the ftraight lines to the
circumferences are equal.

‘ This is not a definition but a theorem, the truth of which
‘ is evident; for, if the circles be applied to one another, fo that
‘ their centres coincide, the circles muft likewife coincide, fince
‘ the ftraight lines from the centres are equal.

II.

A ftraight line is faid to touch
a circle, when it meets the
circle, and being produced
does not cut it.

III.

Circles are faid to touch one
another, which meet, but
do not cut one another.
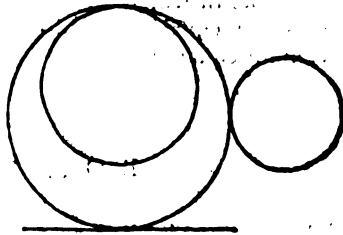
IV.

Straight lines are faid to be equally di-
ftant from the centre of a circle,
when the perpendiculars drawn to
them from the centre are equal.

V.

And the ftraight line on which the
greater perpendicular falls, is faid to
be farther from the centre.
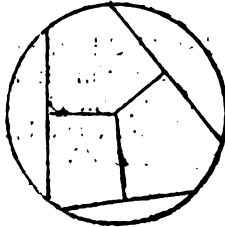
E 2   VI.

VI.

A fegment of a circle is the figure con-
tained by a ftraight line and the cir-
cumference it cuts off.

VII.

" The angle of a fegment is that which is contained by the
" ftraight line and the circumference."

VIII.

An angle in a fegment is the angle con-
tained by two ftraight lines drawn
from any point in the circumference,
of the fegment, to the extremities
of the ftraight line which is the bafe
of the fegment.

IX.
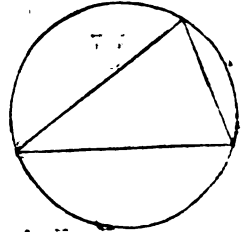
And an angle is faid to infift or ftand
upon the circumference intercepted
between the ftraight lines that con-
tain the angle.

X.

The fector of a circle is the figure contain-
ed by two ftraight lines drawn from the
centre, and the circumference between
them.

XI.
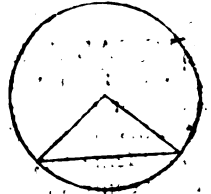
Similar fegments of a circle,
are thofe in which the an-
gles are equal, or which
contain equal angles.

PROP. I. PROB.

To find the centre of a given circle.

Let ABC be the given circle; it is required to find its centre.
Draw within it any ftraight line AB, and bifect it in D;
from the point D draw DC at right angles to AB, and pro-
duce it to E, and bifect CE in F: The point F is the centre of
the circle ABC.

a 10. 1.
b 11. 1.

For,

For, if it be not, let, if poffible, G be the centre, and join GA, GD, GB: Then, becaufe DA is equal to DB, and DG common to the two triangles ADG, BDG, the two fides AD, DG are e-qual to the two BD, DG, each to each; and the bafe GA is equal to the bafe GB, becaufe they are drawn from the centre G *: Therefore the angle ADG is equal ᶜ to the angle GDB: But when a ftraight line ftand-ing upon another ftraight line makes the adjacent angles equal to one ano-ther, each of the angles is a right an-gle ᵈ: Therefore the angle GDB is a right angle: But FDB is likewife a right angle; wherefore the angle FDB is equal to the angle GDB, the greater to the lefs, which is impoffible: Therefore G is not the centre of the cir-cle ABC: In the fame manner it can be fhewn, that no other point but F is the centre; that is, F is the centre of the circle ABC: Which was to be found.
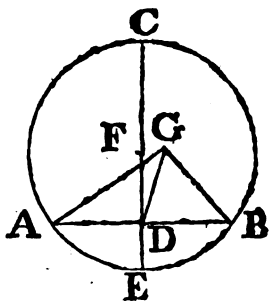
c 8. 1.

d 10. def. 1.

Cor. From this it is manifeft, that if in a circle a ftraight line bifect another at right angles, the centre of the circle is in the line which bifects the other.

PROP. II. THEOR.

IF any two points be taken in the circumference of a circle, the ftraight line which joins them fhall fall within the circle.

Let ABC be a circle, and A, B any two points in the cir-cumference; the ftraight line drawn from A to B fhall fall within the circle.

For, if it do not, let it fall, if poffi-ble, without, as ALB; find * D the cen-tre of the circle ABC, and join AD, DB, and produce DF, any ftraight line meeting the circumference AB, to E: Then becaufe DA is equal to DB, the angle DAB is equal ᵇ to the angle DBA; and becaufe AE, a fide of the triangle DAE,
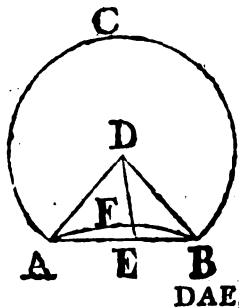
a 1. 3.

b 5. 1.

E 3

DAE,

c 16. 1.

DAE, is produced to B, the angle DEB is greater [c] than the
angle DAE; but DAE is equal to the angle DBE; therefore
the angle DEB is greater than the angle DBE: But to the great-
er angle the greater side is opposite [d]; DB is therefore greater

d 19. 1

than DE : But DB is equal to DF; wherefore DF is greater
than DE, the less than the greater, which is impossible: There-
fore the straight line drawn from A to B does not fall without
the circle. In the same manner, it may be demonstrated that
it does not fall upon the circumference; it falls therefore with-
in it. Wherefore, if any two points, &c. Q. E. D.

PROP. III. THEOR.

IF a straight line drawn through the centre of a circle
bisect a straight line in it which does not pass through
the centre, it shall cut it at right angles; and, if it cuts it
at right angles, it shall bisect it.

Let ABC be a circle, and let CD, a straight line drawn
through the centre, bisect any straight line AB, which does not
pass through the centre, in the point F: It cuts it also at right
angles.

a 1. 3.

Take [a] E the centre of the circle, and join EA, EB: Then,
because AF is equal to FB, and FE common to the two tri-
angles AFE, BFE, there are two sides in the one equal to two
sides in the other, and the base EA is
equal to the base EB; therefore the

b 8. 1.

angle AFE is equal [b] to the Angle BFE:
But when a straight line standing upon
another makes the adjacent angles equal
to one another, each of them is a right

c 10. def. 1.[c]

angle: Therefore each of the angles
AFE, BFE is a right angle; wherefore
the straight line CD, drawn through the
centre bisecting another AB that does
not pass through the centre, cuts the same
at right angles.
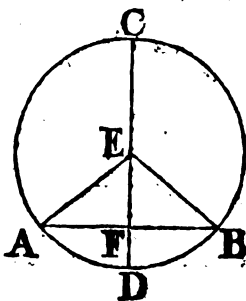
But let CD cut AB at right angles; CD also bisects it, that
is, AF is equal to FB.

The same construction being made, because EA, EB from

d 5. 1.

the centre are equal to one another, the angle EAF is equal [d]
to the angle EBF; and the right angle AFE is equal to the
right angle BFE: Therefore, in the two triangles EAF, EBF,

there

there are two angles in one equal to two angles in the other, **Book III.** and the fide EF, which is oppofite to one of the equal angles in each, is common to both; therefore the other fides are e- qual [c]; AF therefore is equal to FB. Wherefore, if a ftraight e 26. 1. line, &c. Q. E. D.

PROP. IV. THEOR.

IF in a circle two ftraight lines cut one another which do not both pafs through the centre, they do not bifect each the other.

Let ABCD be a circle, and AC, BD two ftraight lines in it which cut one another in the point E, and do not both pafs through the centre: AC, BD do not bifect one another.

For, if it is poffible, let AE be equal to EC, and BE to ED: If one of the lines pafs through the centre, it is plain that it cannot be bifected by the other which does not pafs through the centre: But, if neither of them pafs through the centre, take [a] F the centre of the circle, and join EF: And becaufe FE, a ftraight line through the centre, bifects another AC which does not pafs through the centre, it fhall cut it at right [b] angles; wherefore FEA is a right angle: A- gain, becaufe the ftraight line FE bi- fects the ftraight line BD which does not pafs through the centre, it fhall cut it at right [b] angles; wherefore FEB is a right angle: And FEA was fhewn to be a right angle; therefore FEA is e- qual to the angle FEB, the lefs to the greater, which is im- poffible: Therefore AC, BD do not bifect one another. Where- fore, if in a circle, &c. Q. E. D.

a 1. 3.

b 3. 3.
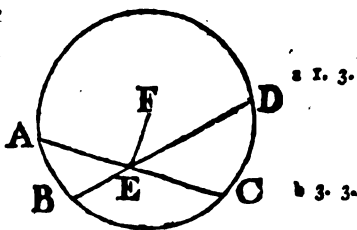
PROP. V. THEOR.

IF two circles cut one another, they fhall not have the fame centre.

Let the two circles ABC, CDG cut one another in the points B, C; they have not the fame centre.

E 4

For,

For, if it be poſſible, let E be their centre : Join EC, and
draw any ſtraight line EFG meet-
ing them in F and G; And becauſe
E is the centre of the circle ABC,
CE is equal to EF : Again, be-
cauſe E is the centre of the circle
CDG, CE is equal to EG : But
CE was ſhewn to be equal to EF;
therefore EF is equal to EG, the
leſs to the greater, which is impoſ-
ſible : Therefore E is not the cen-
tre of the circles ABC, CDG.
Wherefore, if two circles, &c.   Q. E. D.
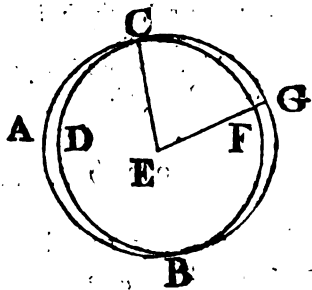
PROP. VI. THEOR.

IF two circles touch one another internally, they ſhall
not have the ſame centre.

Let the two circles ABC, CDE, touch one another internal-
ly in the point C : They have not the ſame centre.
For, if they can, let it be F; join FC and draw any ſtraight
line FEB meeting them in E and B;
And becauſe F is the centre of the
circle ABC, CF is equal to FB : Al-
ſo, becauſe F is the centre of the
circle CDE, CF is equal to FE : And
CF was ſhewn equal to FB ; there-
fore FE is equal to FB, the leſs to
the greater, which is impoſſible :
Wherefore F is not the centre of the
circles ABC, CDE.   Therefore, if
two circles, &c. Q. E. D.
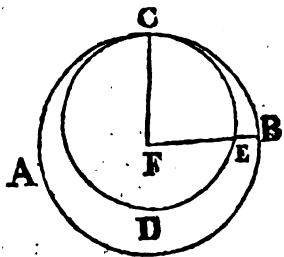
PROP.

P R O P. VII. T H E O R.

IF any point be taken in the diameter of a circle, which
is not the centre, of all the ſtraight lines which can
be drawn from it to the circumference, the greateſt is
that in which the centre is, and the other part of that dia-
meter is the leaſt ; and, of any others, that which is
nearer to the line which paſſes through the centre is al-
ways greater than one more remote : And from the ſame
point there can be drawn only two ſtraight lines that are
equal to one another, one upon each ſide of the ſhorteſt
line.

    Let ABCD be a circle, and AD its diameter, in which let
any point F be taken which is not the centre : Let the centre
be E ; of all the ſtraight lines FB, FC, FG, &c. that can be
drawn from F to the circumference, FA is the greateſt, and
FD, the other part of the diameter AD, is the leaſt ; and of
the others, FB is greater than FC, and FC than FG.
    Join BE, CE, GE ; and becauſe two ſides of a triangle are
greater [a] than the third, BE, EF are greater than BF ; but AE a 20. 1.
is equal to EB ; therefore AE, EF,
that is AF, is greater than BF : A-
gain, becauſe BE is equal to CE,
and FE common to the triangles
BEF, CEF, the two ſides BE, EF
are equal to the two CE, EF ; but
the angle BEF is greater than the
angle CEF ; therefore the baſe BF is
greater [b] than the baſe FC : For the   b 24. 1.
ſame reaſon, CF is greater than GF :
Again, becauſe GF, FE are greater
[a] than EG, and EG is equal to
ED ; GF, FE are greater than ED : Take away the common
part FE, and the remainder GF is greater than the remainder
FD : Therefore FA is the greateſt, and FD the leaſt of all the
ſtraight lines From F to the circumference ; and BF is greater
than CF, and CF than GF.
    Alſo there can be drawn only two equal ſtraight lines from
the point F to the circumference, one upon each ſide of the
                ſhorteſt
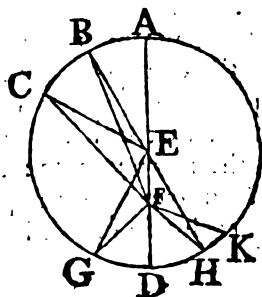
fhorteft line FD : At the point E in the ftraight line EF, make
c the angle FEH equal to the angle GEF, and join FH : Then
becaufe GE is equal to EH, and EF common to the two tri-
angles GEF, HEF; the two fides GE, EF are equal to the two
HE, EF ; and the angle GEF is equal to the angle HEF ; there-
fore the bafe FG is equal d to the bafe FH ; But, befides FH, no
other ftraight line can be drawn from F to the circumference
equal to FG : For, if there can, let it be FK; and becaufe
FK is equal to FG, and FG to FH, FK is equal to FH; that is,
a line nearer to that which paffes through the centre, is equal to
one which is more remote; which is impoffible.　Therefore, if
any point be taken, &c. Q. E. D.

PROP. VIII. THEOR.

IF any point be taken without a circle, and ftraight
lines be drawn from it to the circumference, where-
of one paffes through the centre; of thofe which fall
upon the concave circumference, the greateft is that
which paffes through the centre ; and of the reft, that
which is nearer to that through the centre is always
greater than the more remote : But of thofe which fall
upon the convex circumference, the leaft is that between
the point without the circle, and the diameter ; and of
the reft, that which is nearer to the leaft is always lefs
than the more remote : And only two equal ftraight lines
can be drawn from the point unto the circumference, one
upon each fide of the leaft.

Let ABC be a circle, and D any point without it, from which
let the ftraight lines DA, DE, DF, DC be drawn to the cir-
cumference, whereof DA paffes through the centre.　Of thofe
which fall upon the concave part of the circumference AEFC,
the greateft is AD which paffes through the centre; and the
nearer to it is always greater than the more remote, viz. DE
than DF, and DF than DC : But of thofe which fall upon the
convex circumference HLKG, the leaft is DG between the
point

point D and the diameter AG; and the nearer to it is always less than the more remote, viz. DK than DL, and DL than DH.

Take [a] M the centre of the circle ABC, and join ME, MF, MC, MK, ML, MH: And because AM is equal to ME, add MD to each, therefore AD is equal to EM, MD; but EM, MD are greater [b] than ED; therefore also AD is greater than ED: Again, because ME is equal to MF, and MD common to the triangles EMD, FMD; EM, MD

are equal to FM, MD; but the angle EMD is greater than the angle FMD; therefore the base ED is greater [c] than the base FD: In like manner it may be shewn that FD is greater than CD: Therefore DA is the greatest; and DE greater than DF, and DF than DC: And because MK, KD are greater [b] than MD, and MK is equal to MG, the remainder KD is greater [d] than the remainder GD, that is, GD is less than KD: And because MK, DK are drawn to the point K within the triangle MLD from M, D, the extremities of its side MD; MK, KD are less [e] than ML, LD, whereof MK is equal

a 1. 3.

b 20. 1.
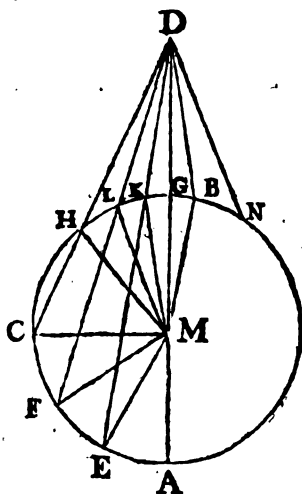
c 24. 1.

d 4. Ax.

e 21. 1.

to ML; therefore the remainder DK is less than the remainder DL: In like manner it may be shewn, that DL is less than DH: Therefore DG is the least, and DK less than DL, and DL than DH: Also there can be drawn only two equal straight lines from the point D to the circumference, one upon each side of the least: At the point M, in the straight line MD, make the angle DMB equal to the angle DMK, and join DB: And because MK is equal to MB, and MD common to the triangles KMD, BMD, the two sides KM, MD are equal to the two BM, MD; and the angle KMD is equal to the angle BMD; therefore the base DK is equal [f] to the base DB: But, besides DB, there can be no straight line drawn from D to the circumference equal to DK: For, if there can, let it be DN; and because DK is equal to DN, and also to DB; therefore DB is equal to DN, that is, the nearer to the least equal to the more remote, which is impossible. If, therefore, any point, &c. Q. E. D.

f 4. 1.

PROP.

PROP. IX. THEOR.

IF a point be taken within a circle, from which there fall more than two equal ftraight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC, from which to the circumference there fall more than two equal ftraight lines, viz. DA, DB, DC, the point D is the centre of the circle.

For, if not, let E be the centre, join DE and produce it to the circumference in F, G; then FG is a diameter of the circle ABC: And becaufe in FG, the diameter of the circle ABC, there is taken the point D which is not the centre, DG fhall be the greateft line from it to the circumference, and DC greater
a 7. 3. * than DB, and DB than DA: But they are likewife equal, which is impoffible: Therefore E is not the centre of the circle ABC: In like manner, it may be demonftrated, that no other point but D is the centre; D therefore is the centre. Wherefore, if a point be taken, &c. Q. E. D.
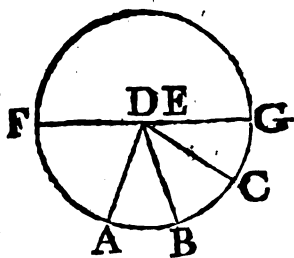
PROP. X. THEOR.

ONE circumference of a circle cannot cut another in more than two points.

If it be poffible, let the circumference ABC cut the circumference DEF in more than two points, viz. in B, G, F; take the centre K of the circle ABC, and join KB, KG, KF: And becaufe within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal ftraight
b 9. 3. lines KB, KG, KF, the point K is b
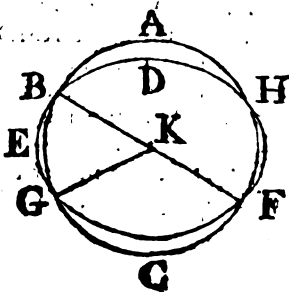
the centre of the circle DEF: But K is alſo the centre of the **Book III.**
circle ABC; therefore the ſame point is the centre of two cir-
cles that cut one another, which is impoſſible [b]. Therefore one [b] 5. 3.
circumference of a circle cannot cut another in more than two
points. Q. E. D.

PROP. XI. THEOR.

IF two circles touch each other internally, the ſtraight
line which joins their centres being produced ſhall paſs
through the point of contact.

Let the two circles ABC, ADE touch each other internally
in the point A, and let F be the centre of the circle ABC,
and G the centre of the circle ADE:
The ſtraight line which joins the
centres F, G, being produced, paſſes
through the point A.
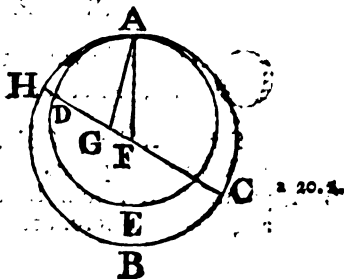
For, if not, let it fall otherwiſe, if
poſſible, as FGDH, and join AF,
AG: And becauſe AG, GF are great-
er [a] than FA, that is, than FH, for [a] 20. 1.
FA is equal to FH, both being from
the ſame centre; take away the com-
mon part FG; therefore the remain-
der AG is greater than the remainder GH: But AG is equal
to GD; therefore GD is greater than GH, the leſs than the
greater, which is impoſſible. Therefore the ſtraight line which
joins the points F, G cannot fall otherwiſe than upon the point
A, that is, it muſt paſs through it. Therefore, if two circles,
&c. Q. E. D.

PROP. XII. THEOR.

IF two circles touch each other externally, the ſtraight
line which joins their centres ſhall paſs through the
point of contact.

Let the two circles ABC, ADE touch each other externally
in the point A; and let F be the centre of the circle ABC, and
G the centre of ADE: The ſtraight line which joins the points
F, G ſhall paſs through the point of contact A.
For, if not, let it paſs otherwiſe, if poſſible, as FCDG, and
join

Book III.
• 20. 1.

join FA, AG : And becaufe F is the centre of the circle ABC, AF is equal to FC : Alfo, becaufe G is the centre of the circle ADE, AG is e-qual to GD : Therefore FA, AG are equal to FC, DG; wherefore the whole FG is greater than FA, AG : But it is alfo lefs a; which is impoffible : Therefore the ftraight line which joins the points F, G fhall not pafs otherwife than through the point of contact A, that is, it muft pafs through it. Therefore, if two circles, &c. Q. E. D.

PROP. XIII. THEOR.

See N.

ONE circle cannot touch another in more points than one, whether it touches it on the infide or outfide.

a 10. 11. 1.

For, if it be poffible, let the circle EBF touch the circle ABC in more points than one, and firft on the infide, in the points B, D ; join BD, and draw a GH bifecting BD at right angles : Therefore, becaufe the points B, D are in the circumfe-

b 2. 3.
c Cor. 1. 3.
d 11. 3.
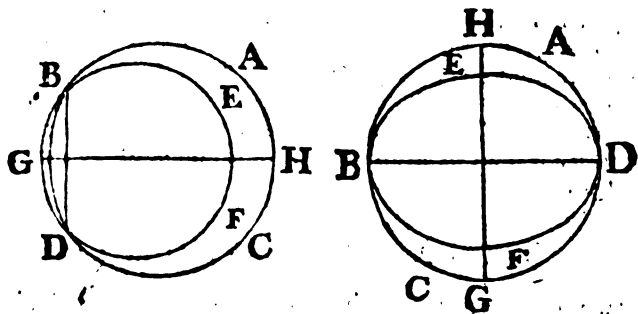
rence of each of the circles, the ftraight line BD falls within b each of them : And their centres are c in the ftraight line GH which bifects BD at right angles; therefore GH paffes through the point of contact d; but it does not pafs through it, becaufe the points B, D are without the ftraight line GH, which is abfurd : Therefore one circle cannot touch another on the infide in more points than one.

Nor can two circles touch one another on the outfide in
more

more than one point: For, if it be poffible, let the circle ACK touch the circle ABC in the points A, C, and join AC: There-fore, becaufe the two points A, C are in the circumference of the circle ACK, the ftraight line AC which joins them fhall fall within ᵇ the circle ACK: And the circle ACK is without the cir-cle ABC; and therefore the ftraight line AC is without this laft circle ; but, be-caufe the points A, C are in the circum-ference of the circle ABC, the ftraight line AC muft be within ᵇ the fame cir-cle, which is abfurd: Therefore one circle cannot touch another on the out-fide in more than one point: And it has been fhewn, that they cannot touch on the infide in more points than one: Therefore, one circle, &c. Q. E. D.
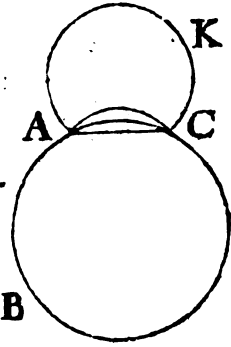
b 2. 3.

PROP. XIV. THEOR.

EQUAL ftraight lines in a circle are equally diftant from the centre; and thofe which are equally di-ftant from the centre, are equal to one another.

Let the ftraight lines AB, CD, in the circle ABDC, be equal to one another; they are equally diftant from the centre.

Take E the centre of the circle ABDC, and from it draw EF, EG perpendiculars to AB, CD : Then, becaufe the ftraight line EF, paffing through the centre, cuts the ftraight line AB, which does not pafs through the centre, at right angles, it alfo bifects ᵃ it: Wherefore AF is equal to FB, and AB double of AF.   For the fame reafon, CD is dou-ble of CG : And AB is equal to CD; therefore AF is equal to CG : And becaufe AE is equal to EC, the fquare of AE is equal to the fquare of EC : But the fquares of AF, FE are equal ᵇ to the fquare of AE, becaufe the angle AFE is a right angle; and, for the like reafon, the fquares of EG, GC are equal to the fquare of EC : Therefore the fquares of AF, FE are equal to the fquares of CG, GE, of which
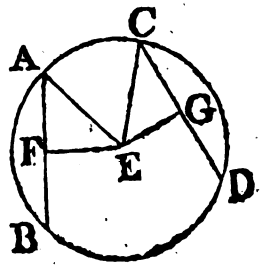
a 3. 3.

b 47. 1.

Book III. which the fquare of AF is equal to the fquare of CG, becaufe
AF is equal to CG; therefore the remaining fquare of FE is
equal to the remaining fquare of EG, and the ftraight line FE
is therefore equal to EG : But ftraight lines in a circle are faid
to be equally diftant from the centre, when the perpendiculars
c 4. Def. 3. drawn to them from the centre are equal ᶜ : Therefore AB, CD
are equally diftant from the centre.

Next, if the ftraight lines AB, CD be equally diftant from
the centre, that is, if FE be equal to EG ; AB is equal to CD :
For, the fame conftruction being made, it may, as before, be de-
monftrated, that AB is double of AF, and CD double of CG, and
that the fquares of EF, FA are equal to the fquares of EG,
GC ; of which the fquare of FE is equal to the fquare of EG,
becaufe FE is equal to EG ; therefore the remaining fquare
of AF is equal to the remaining fquare of CG ; and the ftraight
line AF is therefore equal to CG : And AB is double of AF,
and CD double of CG; wherefore AB is equal to CD. There-
fore equal ftraight lines, &c. Q. E. D.

PROP. XV. THEOR.

THE diameter is the greateft ftraight line in a circle;
and, of all others, that which is nearer to the cen-
tre is always greater than one more remote ; and the
greater is nearer to the centre than the lefs.

Let ABCD de a circle, of which
the diameter is AD, and centre E ;
and let BC be nearer to the centre than
FG ; AD is greater than any ftraight
line BC which is not a diameter, and
BC greater than FG.

From the centre draw EH, EK per-
pendiculars to BC, FG, and join EB,
EC, EF ; and becaufe AE is equal to
EB, and ED to EG, AD is equal to
a 20. I. EB, EC : But EB, EC, are greaterᵃ
than BC ; wherefore, alfo AD is great-
er than BC.
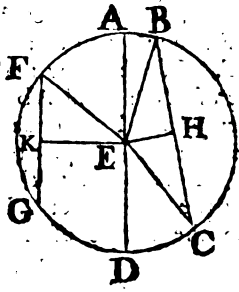
And, becaufe BC is nearer to the centre than FG,
EH

EH is less ᵇ than EK : But, as was demonstrated in the pre- Book III.
ceding, BC is double of BH, and FG double of FK, and the
squares of EH, HB are equal to the squares of EK, KF, of ᵇ 5. Def. 3.
which the square of EH is less than the square of EK, because
EH is less than EK ; therefore the square of BH is greater than
the square of FK, and the straight line BH greater than FK ;
and therefore BC is greater than FG.

Next, Let BC be greater than FG ; BC is nearer to the cen-
tre than FG, that is, the same construction being made, EH is
less than EK : Because BC is greater than FG, BH likewise is
greater than FK : And the squares of BH, HE are equal to
the squares of FK, KE, of which the square of BH is greater
than the square of FK, because BH is greater than FK ; there-
fore the square of EH is less than the square of EK, and the
straight line EH less than EK.   Wherefore the diameter, &c.
Q. E. D.

P R O P. XVI.   T H E O R.

THE straight line drawn at right angles to the dia- See N.
meter of a circle, from the extremity of it, falls
without the circle ; and no straight line can be drawn
between that straight line and the circumference from
the extremity, so as not to cut the circle ; or, which is
the same thing, no straight line can make so great an a-
cute angle with the diameter at its extremity, or so small
an angle with the straight line which is at right angles to
it, as not to cut the circle.

Let ABC be a circle, the centre of which is D, and the dia-
meter AB ; the straight line drawn at right angles to AB from
its extremity A, shall fall without the circle.

For, if it does not, let it fall, if
possible, within the circle as AC,
and draw DC to the point C where
it meets the circumference : And
because DA is equal to DC, the
angle DAC is equal ᵃ to the angle
ACD ; but DAC is a right angle,
therefore ACD is a right angle,
and the angles DAC, ACD are
therefore equal to two right angles ; which is impossible ᵇ : ᵇ 17. 1.
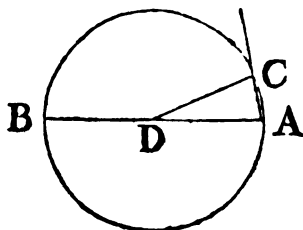
a 5. 1.

F   Therefore

Therefore the ftraight line drawn from A at right angles to BA
does not fall within the circle: In the fame manner, it may be
demonftrated that it does not fall upon the circumference; there-
fore it muft fall within the circle, as AE.

And between the ftraight line AE and the circumference no
ftraight line can be drawn from the point A which does not
cut the circle: For, if poffible, let FA be between them, and
from the point D draw c DG perpendicular to FA, and let it
meet the circumference in H: And becaufe AGD is a right
angle, and DAG lefs b than a right angle: DA is greater d than
DG: But DA is equal to DH;
therefore DH is greater than DG,
the lefs than the greater, which is
impoffible: Therefore no ftraight
line can be drawn from the point
A between AE and the circumfe-
rence, which does not cut the cir-
cle, or, which amounts to the fame
thing, however great an acute angle
a ftraight line makes with the dia-
meter at the point A, or however
fmall an angle it makes with AE,
the circumference paffes between that ftraight line and the per-
pendicular AE. ' And this is all that is to be underftood,
' when, in the Greek text and tranflations from it, the angle of
' the femicircle is faid to be greater than any acute rectilineal
' angle, and the remaining angle lefs than any rectilineal an-
' gle.'
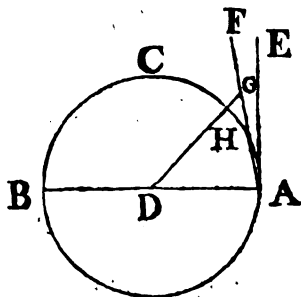
COR. From this it is manifeft that the ftraight line which is
drawn at right angles to the diameter of a circle from the ex-
tremity of it, touches the circle; and that it touches it only in
one point, becaufe, if it did meet the circle in two, it would
fall within it e. ' Alfo it is evident that there can be but one
' ftraight line which touches the circle in the fame point.'

c 12. 1.

d 19. 1.

e 2. 3.

PROP. XVII.   PROB.

TO draw a ftraight line from a given point, either
without or in the circumference, which fhall touch a
given circle.

Firft, Let A be a given point without the given circle BCD;

it is required to draw a ftraight line from A which fhall touch the circle.

Find [a] the centre E of the circle, and join AE; and from the centre E, at the diftance EA, defcribe the circle AFG; from the point D draw [b] DF at right angles to EA, and join EBF, AB; AB touches the circle BCD.

Becaufe E is the centre of the circles BCD, AFG, EA is equal to EF: And ED to EB; therefore the two fides AE, EB are equal to the two FE, ED, and they contain the angle at E common to the two triangles AEB, FED; therefore the bafe DF is equal to the bafe AB, and the triangle FED to the triangle

AEB, and the other angles to the other angles [c]: Therefore the angle EBA is equal to the angle EDF: But EDF is a right angle, wherefore EBA is a right angle: And EB is drawn from the centre; but a ftraight line drawn from the extremity of a diameter, at right angles to it, touches the circle [d]: Therefore AB touches the circle; and it is drawn from the given point A. Which was to be done.
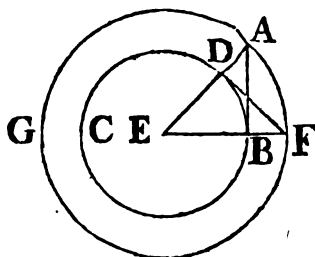
But, if the given point be in the circumference of the circle, as the point D, draw DE to the centre E, and DF at right angles to DE; DF touches the circle [d].

PROP. XVIII. THEOR.

IF a ftraight line touches a circle, the ftraight line drawn from the centre to the point of contact, fhall be perpendicular to the line touching the circle.

Let the ftraight line DE touch the circle ABC in the point C, take the centre F, and draw the ftraight line FC; FC is perpendicular to DE.

For, if it be not, from the point F draw FBG perpendicular to DE; and becaufe FGC is a right angle, GCF is [b] an acute angle; and to the greater angle the greateft [c] fide is oppofite:

Therefore

Therefore FC is greater than FG ;
but FC is equal to FB ; therefore
FB is greater than FG, the lefs than
the greater, which is impoffible :
Wherefore FG is not perpendicular
to DE : In the fame manner it may
be fhewn, that no other is perpen-
dicular to it befides FC, that is, FC
is perpendicular to DE. Therefore,
if a ftraight line, &c. Q. E. D.
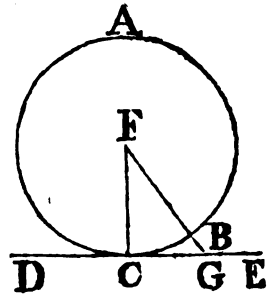
PROP. XIX. THEOR.

IF a ftraight line touches a circle, and from the point
of contact a ftraight line be drawn at right angles to
the touching line, the centre of the circle fhall be in
that line.

Let the ftraight line DE touch the circle ABC in C, and
from C let CA be drawn at right angles to DE ; the centre of
the circle is in CA.

For, if not, let F be the centre, if poffible, and join CF :
Becaufe DE touches the circle ABC,
and FC is drawn from the centre to
the point of contact, FC is perpen-
dicular[a] to DE ; therefore FCE is a
right angle : But ACE is alfo a right
angle ; therefore the angle FCE is
equal to the angle ACE, the lefs to
the greater, which is impoffible :
Wherefore F is not the centre of the
circle ABC : In the fame manner, it
may be fhewn, that no other point
which is not in CA, is the centre ;
that is, the centre is in CA. Therefore, if a ftraight line, &c.
Q. E. D.
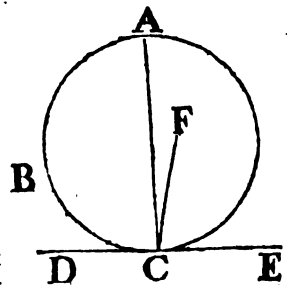
PROP. XX. THEOR.

THE angle at the centre of a circle is double of the
angle at the circumference, upon the fame bafe,
that is, upon the fame part of the circumference.

Let

. Let ABC be a circle, and BEC an angle at the centre, and
BAC an angle at the circumference, which have the same cir-
cumference BC for their base ; the angle
BEC is double of the angle BAC.

Firſt, Let E the centre of the circle be
within the angle BAC, and join AE, and
produce it to F: Becauſe EA is equal
to EB, the angle EAB is equal ᵃ to
the angle EBA ; therefore the angles
EAB, EBA are double of the angle EAB ;
but the angle BEF is equal ᵇ to the angles
EAB, EBA ; therefore alſo the angle
BEF is double of the angle EAB: For the
ſame reaſon, the angle FEC is double of
the angle EAC : Therefore the whole angle BEC is double of
the whole angle BAC.
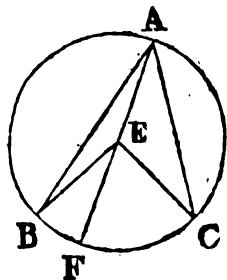
a 5. 1.

b 32. 1.

Again, Let E the centre of the
circle be without the angle BDC, and
join DE and produce it to G. It
may be demonſtrated, as in the firſt
caſe, that the angle GEC is double
of the angle GDC, and that GEB a
part of the firſt is double of GDB a
part of the other; therefore the re-
maining angle BEC is double of the
remaining angle BDC. Therefore
the angle at the centre, &c. Q. E. D.
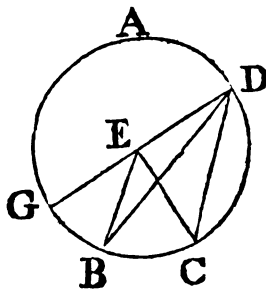
PROP. XXI. THEOR.

THE angles in the ſame ſegment of a circle are equal
to one another.

Let ABCD be a circle, and BAD,
BED angles in the ſame ſegment
BAED: The angles BAD, BED are
equal to one another.

Take F the centre of the circle
ABCD: And, firſt, let the ſegment
BAED be greater than a ſemicircle,
and join BF, FD : And becauſe the
angle BFD is at the centre, and the
angle BAD at the circumference,
and that they have the ſame part of

F 3

the
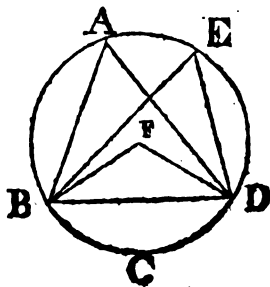
the circumference, viz. BCD, for their base; therefore the an-
gle BFD is double ª of the angle BAD : For the same reason,
the angle BFD is double of the angle BED : Therefore the
angle BAD is equal to the angle BED.

But, if the segment BAED be not greater than a semicircle,
let BAD, BED be angles in it; these
also are equal to one another : Draw
AF to the centre, and produce it to
C, and join CE : Therefore the seg-
ment BADC is greater than a semi-
circle; and the angles in it BAC,
BEC are equal, by the first case : For
the same reason, because CBED is
greater than a semicircle, the angles
CAD, CED are equal : Therefore
the whole angle BAD is equal to the
whole angle BED. Wherefore the angles in the same segment,
&c. Q. E. D.
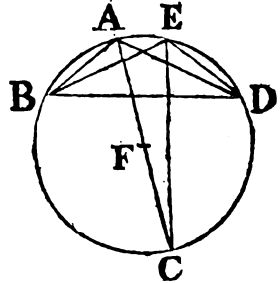
PROP. XXII.   THEOR.

THE opposite angles of any quadrilateral figure de-
scribed in a circle, are together equal to two right
angles.

Let ABCD be a quadrilateral figure in the circle ABCD;
any two of its opposite angles are together equal to two right
angles.

Join AC, BD; and because the three angles of every tri-
angle are equal ª to two right angles, the three angles of the
triangle CAB, viz. the angles CAB, ABC, BCA are equal to
two right angles : But the angle CAB
is equal ᵇ to the angle CDB, because
they are in the same segment BADC;
and the angle ACB is equal to the
angle ADB, because they are in the
same segment ADCB : Therefore the
whole angle ADC is equal to the
angles CAB, ACB : To each of these
equals add the angle ABC; therefore
the angles ABC, CAB, BCA are e-
qual to the angles ABC, ADC : But ABC, CAB, BCA are
equal to two right angles; therefore also the angles ABC, ADC
are equal to two right angles : In the same manner, the angles
BAD,
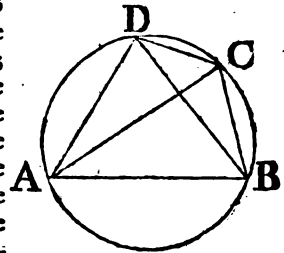
BAD, DCB may be fhewn to be equal to two right angles.
Therefore, the oppofite angles, &c. Q. E. D.

PROP. XXIII. THEOR.

UPON the fame ftraight line, and upon the fame fide See N.
of it, there cannot be two fimilar fegments of
circles, not coinciding with one another.

If it be poffible, let the two fimilar fegments of circles, viz.
ACB, ADB, be upon the fame fide of the fame ftraight line
AB, not coinciding with one another: Then, becaufe the cir-
cle ACB cuts the circle ADB in the
two points A, B, they cannot cut one
another in any other point ª: One of ª 10. 3.
the fegments muft therefore fall within
the other; let ACB fall within ADB,
and draw the ftraight line BCD, and
join CA, DA: And becaufe the feg-
ment ACB is fimilar to the fegment
ADB, and that fimilar fegments of circles contain ᵇ equal an- ᵇ 11. def. 3.
gles; the angle ACB is equal to the angle ADB, the exterior
to the interior, which is impoffible ᶜ. Therefore, there cannot ᶜ 16. 1.
be two fimilar fegments of a circle upon the fame fide of the
fame line, which do not coincide. Q. E. D.
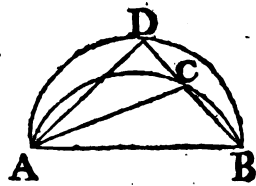
PROP. XXIV. THEOR.

SIMILAR fegments of circles upon equal ftraight See N.
lines, are equal to one another.

Let AEB, CFD be fimilar fegments of circles upon the equal
ftraight lines AB, CD; the fegment AEB is equal to the feg-
ment CFD.
For, if the feg-
ment AEB be
applied to the
fegment CFD,
fo as the point A
be on C, and
the ftraight line
AB upon CD, the point B fhall coincide with the point D, be-
E 4   caufe
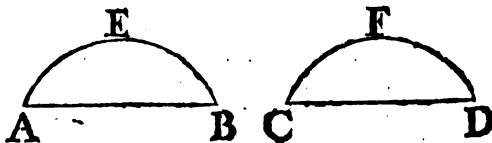
caufe AB is equal to CD : Therefore the ſtraight line AB co-
inciding with CD, the ſegment AEB muſt [a] coincide with the
ſegment CFD, and therefore is equal to it.   Wherefore ſimilar
ſegments, &c. Q. E. D.

PROP. XXV. PROB.

A Segment of a circle being given, to deſcribe the
circle of which it is the ſegment.

Let ABC be the given ſegment of a circle; it is required to
deſcribe the circle of which it is the ſegment.

Biſect [a] AC in D, and from the point D draw [b] DB at right
angles to AC, and join AB : Firſt, let the angles ABD, BAD,
be equal to one another; then the ſtraight line BD is equal [c]
to DA, and therefore to DC ; and becauſe the three ſtraight
lines DA, DB, DC are all equal, D is the centre of the cir-
cle [d] : From the centre D, at the diſtance of any of the three
DA, DB, DC, deſcribe a circle ; this ſhall paſs through the other
points ; and the circle of which ABC is a ſegment is deſcribed :
And becauſe the centre D is in AC, the ſegment ABC is a ſe-

micircle : But if the angles ABD, BAD are not equal to one
another, at the point A, in the ſtraight line AB, make [e] the angle
BAE equal to the angle ABD, and produce BD, if neceſſary, to E,
and join EC : And becauſe the angle ABE is equal to the angle
BAE, the ſtraight line BE is equal [e] to EA : And becauſe AD
is equal to DC, and DE common to the triangles ADE, CDE,
the two ſides AD, DE are equal to the two CD, DE, each to
each ; and the angle ADE is equal to the angle CDE, for
each of them is a right angle ; therefore the baſe AE is equal
[f] to the baſe EC : But AE was ſhewn to be equal to EB, where-
fore alſo BE is equal to EC : And the three ſtraight lines AE,
                                                              EB,
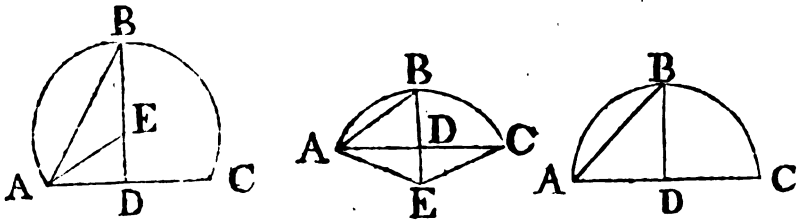
EB, EC are therefore equal to one another; wherefore ᵈ E is the centre of the circle. From the centre E, at the diſtance of any of the three AE, EB, EC, deſcribe a circle, this ſhall paſs ᵈ 9. 3. through the other points; and the circle of which ABC is a ſegment is deſcribed : And it is evident that if the angle ABD be greater than the angle BAD, the centre E falls without the ſegment ABC, which therefore is leſs than a ſemicircle : But if the angle ABD be leſs than BAD, the centre E falls within the ſegment ABC, which is therefore greater than a ſemicircle : Wherefore a ſegment of a circle being given, the circle is deſcribed of which it is a ſegment.   Which was to be done.

PROP. XXVI.   THEOR.

IN equal circles, equal angles ſtand upon equal circumferences, whether they be at the centres or circumferences.

Let ABC, DEF be equal circles, and the equal angles BGC, EHF at their centres, and BAC, EDF at their circumferences : The circumference BKC is equal to the circumference ELF.

Join BC, EF ; and becauſe the circles ABC, DEF are equal, the ſtraight lines drawn from their centres are equal : Therefore the two ſides BG, GC, are equal to the two EH, HF ;
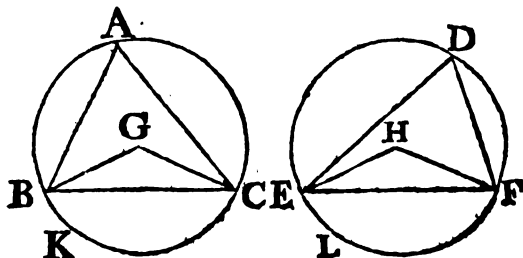
and the angle at G is equal to the angle at H ; therefore the baſe BC is equal ᵃ to the baſe EF : And becauſe the angle at A ᵃ 4. 1. is equal to the angle at D, the ſegment BAC is ſimilar ᵇ to the ᵇ 11. def. 3. ſegment EDF ; and they are upon equal ſtraight lines BC, EF ; but ſimilar ſegments of circles upon equal ſtraight lines are equal ᶜ to one another ; therefore the ſegment BAC is equal ᶜ 24. 3. to the ſegment EDF : But the whole circle ABC is equal to the
whole

whole DEF; therefore the remaining fegment BKC is equal to the remaining fegment ELF, and the circumference BKC to the circumference ELF. Wherefore, in equal circles, &c. Q. E. D.

PROP. XXVII.   THEOR.

IN equal circles, the angles which ftand upon equal circumferences, are equal to one another, whether they be at the centres or circumferences.

Let the angles BGC, EHF at the centres, and BAC, EDF at the circumferences of the equal circles ABC, DEF ftand upon the equal circumferences BC, EF: The angle BGC is e-qual to the angle EHF, and the angle BAC to the angle EDF.

If the angle BGC be equal to the angle EHF, it is manifeft
a 20. 3.   ᵃ that the angle BAC is alfo equal to EDF: But, if not, one
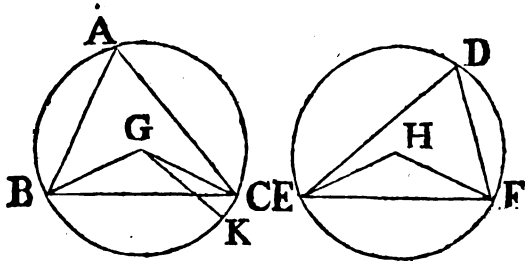
of them is the greater: Let BGC be the greater, and at the
b 23. 1.   point G, in the ftraight line BG, make ᵇ the angle BGK equal to the angle EHF; but equal angles ftand upon equal circum-
c 26. 3.   ferences ᶜ, when they are at the centre; therefore the circum-ference BK is equal to the circumference EF: But EF is equal to BC; therefore alfo BK is equal to BC, the lefs to the great-er, which is impoffible: Therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it: And the angle at A is half of the angle BGC, and the angle at D half of the an-gle EHF: Therefore the angle at A is equal to the angle at D. Wherefore, in equal circles, &c. Q. E D.

                                          PROP.

PROP. XXVIII. THEOR.

IN equal circles, equal ftraight lines cut off equal cir-
cumferences, the greater equal to the greater, and the
lefs to the lefs.

Let ABC, DEF be equal circles, and BC, EF equal ftraight
lines in them, which cut off the two greater circumferences
BAC, EDF, and the two lefs BGC, EHF: The greater BAC
is equal to the greater EDF, and the lefs BGC to the lefs
EHF.

Take *K, L the centres of the circles, and join BK, KC, a 1. 3.
EL, LF: And becaufe the circles are equal, the ftraight lines
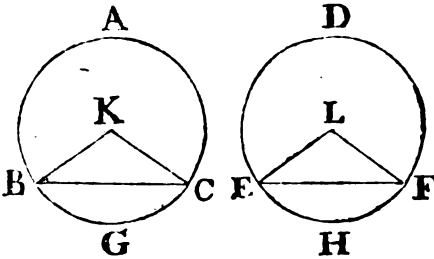
from their centres are equal; therefore BK, KC are equal to
EL, LF; and the bafe BC is equal to the bafe EF; therefore
the angle BKC is equal b to the angle ELF: But equal angles b 8. 1.
ftand upon equal c circumferences, when they are at the cen- c 26. 3.
tres; therefore the circumference BGC is equal to the cir-
cumference EHF. But the whole circle ABC is equal to the
whole EDF; the remaining part therefore of the circumfe-
rence, viz. BAC, is equal to the remaining part EDF. There
fore, in equal circles, &c. Q. E. D.

PROP. XXIX. THEOR.

IN equal circles equal circumferences are fubtended by
equal ftraight lines.

Let ABC, DEF be equal circles, and let the circumferences
BGC, EHF alfo be equal; and join BC, EF: The ftraight line
BC is equal to the ftraight line EF.

Take

Take ᵃ K, L the centres of the circles, and join BK, KC, EL, LF: And becaufe the circumference BGC is equal to the
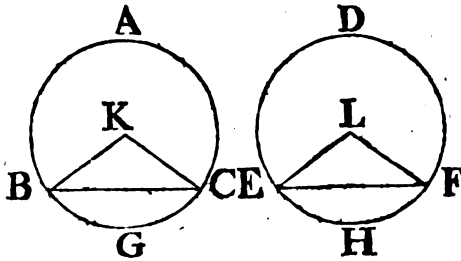
b 27. 3.

circumference EHF, the angle BKC is equal ᵇ to the angle ELF: And becaufe the circles ABC, DEF are equal, the ftraight lines from their centres are equal: Therefore BK, KC are e-qual to EL, LF, and they contain equal angles: Therefore the

c 4. 1.

bafe BC is equal ᶜ to the bafe EF. Therefore, in equal circles, &c. Q. E. D.

P R O P. XXX. P R O B.

TO bifect a given circumference, *that is, to divide it into two equal parts.*

' Let ADB be the given circumference; it is required to bi-fect it.

a 10. 1.

Join AB, and bifect ᵃ it in C; from the point C draw CD at right angles to AB, and join AD, DB: The circumference ADB is bifected in the point D.

Becaufe AC is equal to CB, and CD common to the tri-angles ACD, BCD, the two fides AC, CD are equal to the two BC, CD; and the angle ACD is equal to the angle BCD, becaufe each of them is a right angle; therefore the bafe AD

b 4. 1.

is equal ᵇ to the bafe BD: But equal
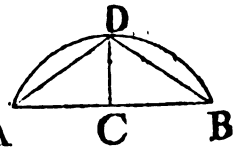
c 28. 3.

ftraight lines cut off equal ᶜ circumferences, the greater equal to the greater, and the lefs to the lefs, and AD, DB are each of them lefs than a femicircle; becaufe DC paffes through the

d Cor. 1. 3.

centre ᵈ. Wherefore the circumference AD is equal to the cir-cumference DB: Therefore the given circumference is bifected in D. Which was to be done.

P R O P.

PROP. XXXI. THEOR.

IN a circle, the angle in a femicircle is a right angle; but the angle in a fegment greater than a femicircle is lefs than a right angle; and the angle in a fegment lefs than a femicircle is greater than a right angle.

Let ABCD be a circle, of which the diameter is BC, and centre E; and draw CA dividing the circle into the fegments ABC, ADC, and join BA, AD, DC; the angle in the femicircle BAC is a right angle; and the angle in the fegment ABC, which is greater than a femicircle, is lefs than a right angle; and the angle in the fegment ADC, which is lefs than a femicircle, is greater than a right angle.

Join AE, and produce BA to F; and becaufe BE is equal to EA, the angle EAB is equal ᵃ to EBA; alfo, becaufe AE a 5. 1. is equal to EC, the angle EAC is equal to ECA; wherefore the whole angle BAC is equal to the two angles ABC, ACB: But FAC, the exterior angle of the triangle ABC, is equal ᵇ to the two angles ABC, ACB; therefore the angle BAC is equal to the angle FAC, and each of them is therefore a right ᶜ angle: Wherefore the angle BAC in a femicircle is a right angle.

b 32. 1.
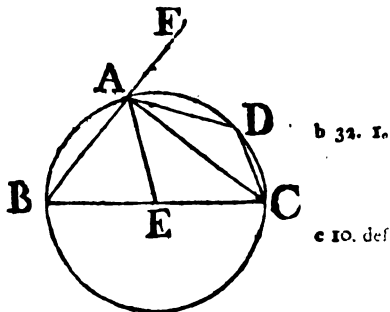
e 10. def

And becaufe the two angles ABC, BAC of the triangle ABC are together lefs ᵈ than two right angles, and that BAC d 17. 1 is a right angle, ABC muft be lefs than a right angle; and therefore the angle in a fegment ABC greater than a femicircle, is lefs than a right angle.

And becaufe ABCD is a quadrilateral figure in a circle, any two of its oppofite angles are equal ᵉ to two right angles; there- e 22. 3. fore the angles ABC, ADC are equal to two right angles; and ABC is lefs than a right angle; wherefore the other ADC is greater than a right angle.

Befides, it is manifeft, that the circumference of the greater fegment ABC falls without the right Angle CAB, but the circumference of the lefs fegment ADC falls within the right angle CAF, ʿ And this is all that is meant, when in the

ʿ Greek

‘ Greek text, and the tranflations from it, the angle of the
‘ greater fegment is faid to be greater, and the angle of the lefs
‘ fegment is faid to be lefs, than a right angle.’

CoR. From this it is manifeft, that if one angle of a tri-
angle be equal to the other two, it is a right angle, becaufe the
angle adjacent to it is equal to the fame two; and when the ad-
jacent angles are equal, they are right angles.

P R O P. XXXII.   T H E O R.

IF a ftraight line touches a circle, and from the point
of contact a ftraight line be drawn cutting the circle,
the angles made by this line with the line touching the
circle, fhall be equal to the angles which are in the alter-
nate fegments of the circle.

Let the ftraight line EF touch the circle ABCD in B, and
from the point B let the ftraight line BD be drawn cutting the
circle : The angles which BD makes with the touching line EF
fhall be equal to the angles in the alternate fegments of the
circle ; that is, the angle FBD is equal to the angle which is in
the fegment DAB, and the angle DBE to the angle in the feg-
ment BCD.

a 11. 1.   From the point B draw a BA at right angles to EF, and
take any point C in the circumference BD, and join AD,
DC, CB ; and becaufe the ftraight line EF touches the circle
ABCD in the point B, and BA is
drawn at right angles to the touch-
ing line from the point of contact
b 19. 3.   B, the centre of the circle is b in BA;
therefore the angle ADB in a femi-
c 31. 3.   circle is a right c angle, and confe-
quently the other two angles BAD,
d 32. 1.   ABD are equal d to a right angle :
But ABF is likewife a right angle ;
therefore the angle ABF is equal to
the angles BAD, ABD : Take from
thefe equals the common angle
ABD ; therefore the remaining an-
gle DBF is equal to the angle BAD, which is in the alternate
fegment of the circle ; and becaufe ABCD is a quadrilateral
e 22. 3.   figure in a circle, the oppofite angles BAD, BCD are equal e to
                                                                    two
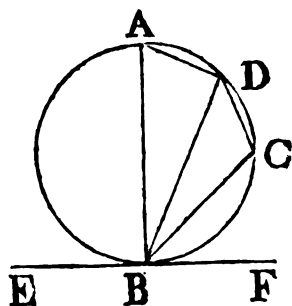
two right angles; therefore the angles DBF, DBE, being like-
wife equal [f] to two right angles, are equal to the angles BAD,
BCD; and DBF has been proved equal to BAD: Therefore the
remaining angle DBE is equal to the angle BCD in the alter-
nate fegment of the circle. Wherefore, if a ftraight line, &c.
Q. E. D.

PROP. XXXIII. PROB.

UPON a given ftraight line to defcribe a fegment of
a circle, containing an angle equal to a given rec-
tilineal angle.

Let AB be the given ftraight line, and the angle at C the
given rectilineal angle; it is required to defcribe upon the given
ftraight line AB a fegment of a circle, containing an angle e-
equal to the angle C.

Firft, Let the angle at C be a
right angle, and bifect [a] AB in F,
and from the centre F, at the di-
ftance FB, defcribe the femicircle
AHB; therefore the angle AHB in
a femicircle is [b] equal to the right
angle at C.
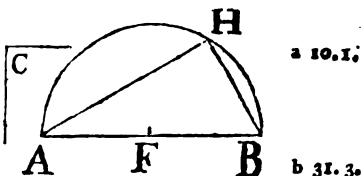
But, if the angle C be not a right angle, at the point A, in
the ftraight line AB, make [c] the angle BAD equal to the angle
C, and from the point A
draw [d] AE at right angles to
AD; bifect [a] AB in F, and
from F draw [d] FG at right
angles to AB, and join GB:
And becaufe AF is equal to
FB, and FG common to the
triangles AFG, BFG, the
two fides AF, FG are equal
to the two BF, FG; and the
angle AFG is equal to the
angle BFG; therefore the
bafe AG is equal [e] to the bafe GB; and the circle defcribed
from the centre G, at the diftance GA, fhall pafs through the
point B; let this be the circle AHB: And becaufe from the
point A the extremity of the diameter AE, AD is drawn at
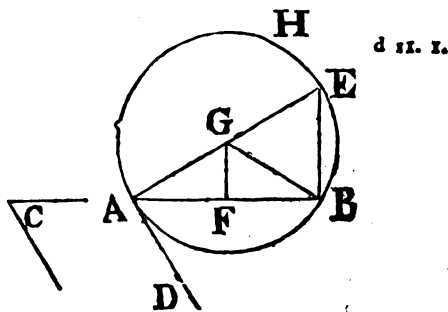
right

right angles to AE, therefore AD f touches the circle; and be-
cause AB drawn from the point

f Cor. 16. 3. of contact A cuts the circle,
the angle DAB is equal to the
angle in the alternate segment
AHB g: But the angle DAB

g 32. 3. is equal to the angle C, there-
fore also the angle C is equal
to the angle in the segment
AHB: Wherefore, upon the
given straight line AB the seg-
ment AHB of a circle is described which contains an angle e-
qual to the given angle at C. Which was to be done.
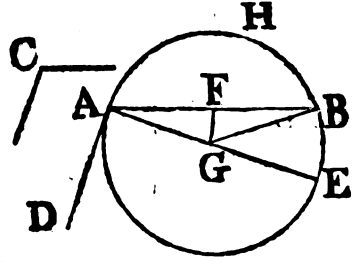
PROP. XXXIV. PROB.

TO cut off a segment from a given circle which shall
contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal
angle; it is required to cut off a segment from the circle ABC
that shall contain an angle equal to the angle D.

a 17. 3. Draw a the straight line EF touching the circle ABC in the
point B, and at the point
B, in the straight line BF,

b 23. 1. make b the angle FBC e-
qual to the angle D:
Therefore, because the
straight line EF touches
the circle ABC, and BC is
drawn from the point of
contact B, the angle FBC

c 32. 3. is equal c to the angle in
the alternate segment BAC
of the circle: But the angle FBC is equal to the angle D;
therefore the angle in the segment BAC is equal to the angle
D: Wherefore the segment BAC is cut off from the given cir-
cle ABC containing an angle equal to the given angle D:
Which was to be done.
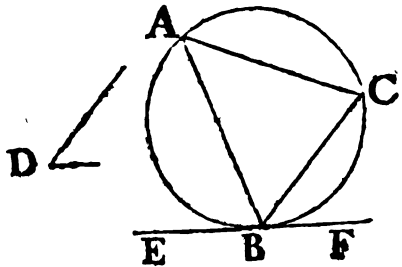
PROP.

P R O P.   XXXV.   T H E O R.

IF two ſtraight lines within a circle cut one another, the See N.
rectangle contained by the ſegments of one of them
is equal to the rectangle contained by the ſegments of
the other.

Let the two ſtraight lines AC, BD, within the circle ABCD,
cut one another in the point E; the rectangle contained by AE,
EC is equal to the rectangle contained by
BE, ED.

If AC, BD paſs each of them through
the centre, ſo that E is the centre; it is
evident, that AE, EC, BE, ED, being all
equal, the rectangle AE, EC is likewiſe
equal to the rectangle BE, ED.
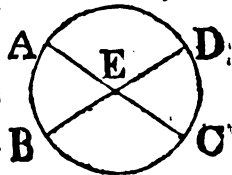
But let one of them BD paſs through the centre, and cut the
other AC, which does not paſs through the centre, at right angles,
in the point E: Then, if BD be biſected in F, F is the centre
of the circle ABCD; and join AF: And becauſe BD, which
paſſes through the centre, cuts the ſtraight line AC, which does
not paſs through the centre, at right
angles in E, AE, EC are equal a to   a 3. 3.
one another: And becauſe the ſtraight
line BD is cut into two equal parts in
the point F, and into two unequal in
the point E, the rectangle BE, ED
together with the ſquare of EF, is e-
qual b to the ſquare of FB; that is,   b 5. 2.
to the ſquare of FA; but the ſquares
of AE, EF are equal c to the ſquare
of FA; therefore the rectangle BE,   c 47. 1.
ED, together with the ſquare of EF,
is equal to the ſquares of AE, EF: Take away the common
ſquare of EF, and the remaining rectangle BE, ED is equal
to the remaining ſquare of AE; that is, to the rectangle AE,
EC.
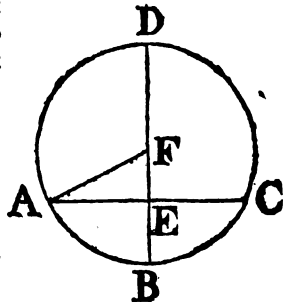
Next, Let BD which paſſes through the centre, cut the other
AC, which does not paſs through the centre, in E, but not at
right angles: Then, as before, if BD be biſected in F, F is the
centre of the circle. Join AF, and from F draw d FG perpen-   d 12. 1.
G   dicular

dicular to AC; therefore AG is equal[a] to GC; wherefore
the rectangle AE, EC, together with the square of EG, is e-
qual[b] to the square of AG : To each of these equals add the
square of GF; therefore the rectangle AE, EC, together with
the squares of EG, GF, is equal to
the squares of AG, GF: But the
squares of EG, GF are equal[c] to the
square of EF; and the squares of
AG, GF are equal to the square of
AF: Therefore the rectangle AE,
EC, together with the square of EF,
is equal to the square of AF; that is,
to the square of FB: But the square
of FB is equal[b] to the rectangle BE, ED, together with the
square of EF; therefore the rectangle AE, EC, together with
the square of EF, is equal to the rectangle BE, ED, together
with the square of EF : Take away the common square of EF,
and the remaining rectangle AE, EC is therefore equal to the
remaining rectangle BE, ED.
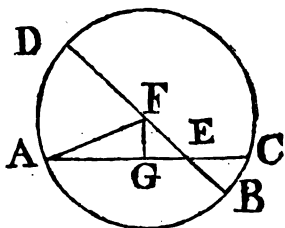
Lastly, Let neither of the straight lines AC, BD pass through
the centre: Take the centre F, and
through E, the intersection of the
straight lines AC, DB, draw the
diameter GEFH : And because the
rectangle AE, EC is equal, as has
been shown, to the rectangle GE,
EH; and, for the same reason, the
rectangle BE, ED is equal to the
same rectangle GE, EH; therefore
the rectangle AE, EC is equal to
the rectangle BE, ED. Wherefore, if two straight lines, &c.
Q. E. D.
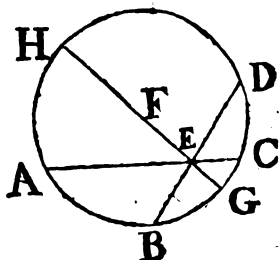
PROP. XXXVI.   THEOR.

IF from any point without a circle two straight lines be
drawn, one of which cuts the circle, and the other
touches it; the rectangle contained by the whole line
which cuts the circle, and the part of it without the cir-
cle, shall be equal to the square of the line which touches
it.

Let D be any point without the circle ABC, and DCA,
DB two straight lines drawn from it, of which DCA cuts the
circle,

tircle, and DB touches the fame: The rectangle AD, DC is equal to the fquare of DB.

Either DCA paffes through the centre, or it does not; firft, let it pafs through the centre E, and join EB; therefore the angle EBD is a right[a] angle: And becaufe the ftraight line AC is bifected in E, and produced to the point D, the rect- angle AD, DC, together with the fquare of EC, is equal[b] to the fquare of ED, and CE is equal to EB: There- fore the rectangle AD, DC, together with the fquare of EB, is equal to the fquare of ED: But the fquare of ED is equal[c] to the fquares of EB, BD, be- caufe EBD is a right angle: Therefore the rectangle AD, DC, together with the fquare of EB, is equal to the fquares of EB, BD: Take away the common fquare of EB; therefore the remaining rectangle AD, DC is equal to the fquare of the tangent DB.
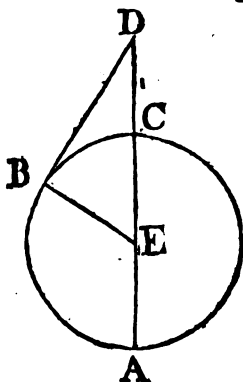
a 18. 3.

b 6. 2.

c 47. 1.

But if DCA does not pafs through the centre of the circle ABC, take[d] the centre E, and draw EF perpendicular[e] to AC, and join EB, EC, ED: And becaufe the ftraight line EF, which paffes through the centre, cuts the ftraight line AC, which does not pafs through the centre, at right angles, it fhall likewife bifect[f] it; there- fore AF is equal to FC: And becaufe the ftraight line AC is bifected in F, and produced to D, the rectangle AD, DC, together with the fquare of FC, is equal[b] to the fquare of FD: To each of thefe equals add the fquare of FE; there- fore the rectangle AD, DC, together with the fquares of CF, FE, is equal to the fquares of DF, FE: But the fquare of ED is equal[c] to the fquares of DF, FE, becaufe EFD is a right angle; and the fquare of EC is equal to the fquares of CF, FE; therefore the rectangle AD, DC, together with the fquare of EC, is equal to the fquare of ED: And CE is equal to EB; therefore the rectangle AD, DC, together with the fquare of EB, is equal to the fquare of ED:

d 1. 3.

e 12. 1.

f 3. 3.

ED:
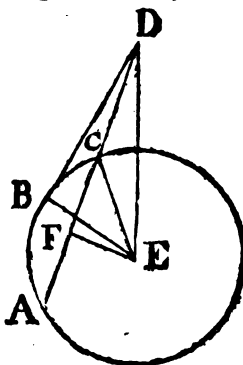
Book III.
c 47. 1.

ED: But the fquares of EB, BD are equal to the fquare* of ED, becaufe EBD is a right angle; therefore the rectangle AD, DC, together with the fquare of EB, is equal to the fquares of EB, BD: Take away the common fquare of EB; therefore the remaining rectangle AD, DC is equal to the fquare of DB. Wherefore, if from any point, &c. Q. E. D.

Cor. If from any point without a circle, there be drawn two ftraight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another, viz. the rectangle BA, AE to the rectangle CA, AF: For each of them is equal to the fquare of the ftraight line AD which touches the circle.
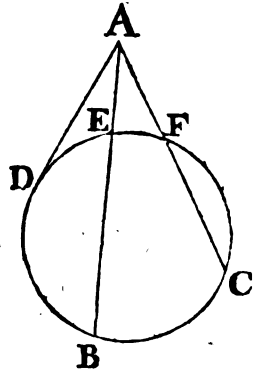
PROP. XXXVII. THEOR.

IF from a point without a circle there be drawn two ftraight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle be equal to the fquare of the line which meets it, the line which meets fhall touch the circle.

Let any point D be taken without the circle ABC, and from it let two ftraight lines DCA and DB be drawn, of which DCA cuts the circle, and DB meets it; if the rectangle AD, DC be equal to the fquare of DB; DB touches the circle.

a 17. 3.
b 18. 3.

c 36. 3.

Draw a the ftraight line DE touching the circle ABC, find its centre F, and join FE, FB, FD; then FED is a right b angle: And becaufe DE touches the circle ABC, and DCA cuts it, the rectangle AD, DC is equal c to the fquare of DE: But the rectangle AD, DC is, by hypothefis, equal to the fquare of DB: Therefore the fquare of DE is equal to the fquare of DB; and the ftraight line DE equal to the ftraight line DB:
And

and FE is equal to FB, wherefore DE, EF are equal to DB, BF; and the bafe FD is common to the two triangles DEF, DBF; therefore the angle DEF is equal ᵈ to the angle DBF; but DEF is a right angle, therefore alfo DBF is a right angle: And FB, if produced, is a diameter, and the ftraight line which is drawn at right angles to a diameter, from the extremity of it, touches ᵉ the circle: Therefore DB touches the circle ABC. Wherefore, if from a point, &c. Q. E. D.

d 8. 1.

e 16. 3.
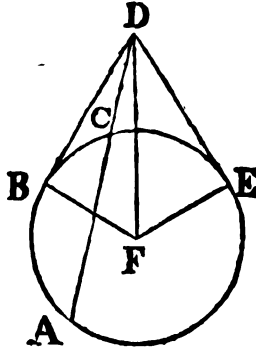
G 3

THE

THE

ELEMENTS

OF

EUCLID.

BOOK IV.

DEFINITIONS.

I.

A Rectilineal figure is said to be inscribed in another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.
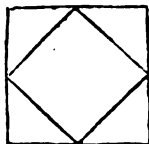
II.

In like manner, a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

III.

A rectilineal figure is said to be inscribed in a circle, when all the angles of the inscribed figure are upon the circumference of the circle.

IV.
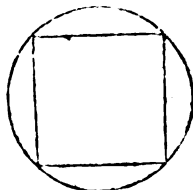
A rectilineal figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.
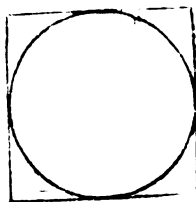
V.

In like manner, a circle is said to be inscribed in a rectilineal figure, when the circumference of the circle touches each side of the figure.

VI.

VI.

A circle is said to be described about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.
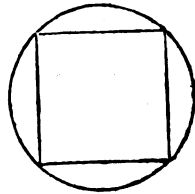
VII.

A straight line is said to be placed in a circle, when the extremities of it are in the circumference of the circle.

PROP. I. PROB.

IN a given circle to place a straight line, equal to a given straight line not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line, not greater than the diameter of the circle.

Draw BC the diameter of the circle ABC; then, if BC is equal to D, the thing required is done; for in the circle ABC a straight line BC is placed equal to D: But, if it is not, BC is greater than D; make CE equal [a] to D, and from the centre C, at the distance CE, describe the circle AEF, and join CA: Therefore, because C is the centre of the circle AEF, CA is equal to CE; but D is equal to CE; therefore D is equal to CA: Wherefore, in the circle ABC, a straight line is placed equal to the given straight line D, which is not greater than the diameter of the circle. Which was to be done.

a 3. 1.
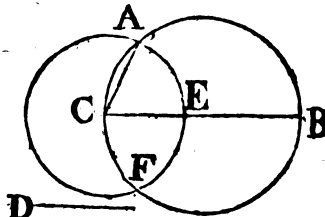
PROP. II. PROB.

IN a given circle to inscribe a triangle equiangular to a given triangle.

G 4

Let

Let ABC be the given circle, and DEF the given triangle ;
it is required to infcribe in the circle ABC a triangle equiangu-
lar to the triangle DEF.

a 17. 3.
Draw ª the ftraight line GAH touching the circle in the
point A, and at the point A, in the ftraight line AH, make
b 23. 1.
ᵇ the angle HAC equal to the angle DEF; and at the point A,
in the ftraight line
AG, make the angle
GAB equal to the
angle DFE, and join
BC : Therefore, be-
caufe HAG touches
the circle ABC, and
AC is drawn from
the point of contact,
the angle HAC is
c 32. 3.
equal ᶜ to the angle
ABC in the alternate fegment of the circle : But HAC is equal
to the angle DEF; therefore alfo the angle ABC is equal to
DEF: For the fame reafon, the angle ACB is equal to the angle
d 32. 1.
DFE; therefore the remaining angle BAC is equal ᵈ to the
remaining angle EDF: Wherefore the triangle ABC is equian-
gular to the triangle DEF, and it is infcribed in the circle ABC.
Which was to be done.
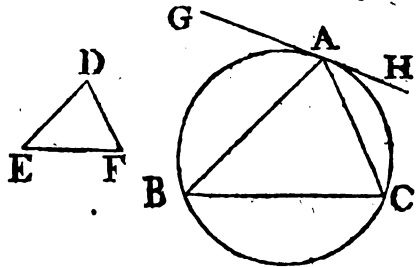
PROP. III.   PROB.

ABOUT a given circle to defcribe a triangle equian-
gular to a given triangle.

Let ABC be the given circle, and DEF the given triangle ;
it is required to defcribe a triangle about the circle ABC equi-
angular to the triangle DEF.

Produce EF both ways to the points G, H, and find the
centre K of the circle ABC, and from it draw any ftraight
a 23. 1.
line KB; at the point K in the ftraight line KB, make ª the
angle BKA equal to the angle DEG, and the angle BKC e-
qual to the angle DFH; and through the points A, B, C, draw
b 17. 3.
the ftraight lines LAM, MBN, NCL touching ᵇ the circle
ABC: Therefore, becaufe LM, MN, NL touch the circle ABC
in the points A, B, C, to which from the centre are drawn
c 18. 3.
KA, KB, KC, the angles at the points A, B, C, are right ᶜ
angles: And becaufe the four angles of the quadrilateral figure
AMBK

AMBK are equal to four right angles, for it can be divided into two triangles; and that two of them KAM, KBM are right

angles, the other two AKB, AMB are equal to two right angles : But the angles DEG, DEF are likewise equal ᵈ to two right angles ; therefore the angles AKB, AMB are equal to the angles DEG, DEF, of which
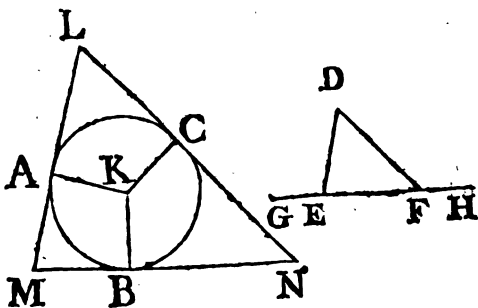
d 23. 1.

AKB is equal to DEG ; wherefore the remaining angle AMB is equal to the remaining angle DEF : In like manner, the angle LNM may be demonſtrated to be equal to DFE ; and therefore the remaining angle MLN is equal ᵉ to the remaining angle EDF ; Wherefore the triangle LMN is equiangular to the triangle DEF : And it is deſcribed about the circle ABC. Which was to be done.

e 32. 1.

PROP. IV. PROB.

To inſcribe a circle in a given triangle.

See N.

Let the given triangle be ABC ; it is required to inſcribe a circle in ABC.

Biſect ᵃ the angles ABC, BCA by the ſtraight lines BD, CD ᵃ 9. 1. meeting one another in the point D, from which draw ᵇ DE, DF, ᵇ 12. 1. DG perpendiculars to AB, BC, CA: And becauſe the angle EBD is equal to the angle FBD, for the angle ABC is biſected by BD, and that the right angle BED, is equal to the right angle BFD, the two triangles EBD, FBD have two angles of the one equal to two angles of the other, and the ſide BD, which is oppoſite to one of the equal angles in each, is common to both ; therefore their other ſides ſhall be equal ᶜ ; where-

c 26. 1.
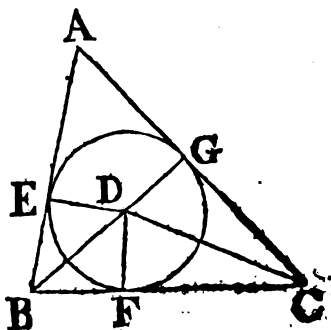
fore

Book IV. fore DE is equal to DF : For the same reason, DG is equal to
DF; therefore the three ftraight lines DE, DF, DG are equal to
one another, and the circle defcribed from the centre D, at the
diftance of any of them, fhall pafs through the extremities of the
other two, and touch the ftraight lines AB, BC, CA, becaufe
the angles at the points E, F, G are right angles, and the
ftraight line which is drawn from the extremity of a diameter
d 16. 3. at right angles to it, touches d the circle : Therefore the ftraight
lines AB, BC, CA do each of them touch the circle, and the
circle EFG is infcribed in the triangle ABC. Which was to
be done.

PROP. V. PROB.

See N. To defcribe a circle about a given triangle.

Let the given triangle be ABC; it is required to defcribe a
circle about ABC.

a 10. 1. Bifect a AB, AC in the points D, E, and from thefe points
b 11. 1. draw DF, EF at right angles b to AB, AC; DF, EF produced
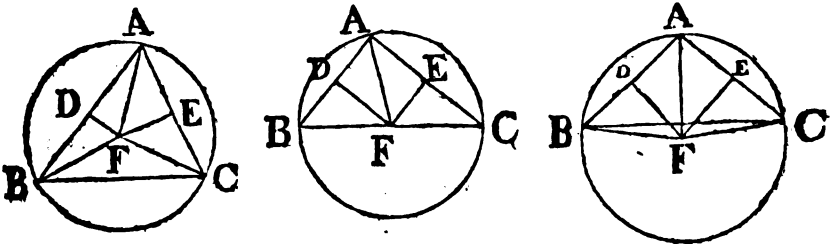
meet one another : For, if they do not meet, they are parallel,
wherefore AB, AC, which are at right angles to them, are pa-
rallel; which is abfurd : Let them meet in F, and join FA ;
alfo, if the point F be not in BC, join BF, CF : Then, becaufe
AD is equal to DB, and DF common, and at right angles to
c 4. 1. AB, the bafe AF is equal c to the bafe FB : In like manner, it
may be fhown that CF is equal to FA; and therefore BF
is equal to FC; and FA, FB, FC are equal to one another ;
wherefore

wherefore the circle defcribed from the centre F, at the di-
ftance of one of them, fhall pafs through the extremities of the
other two; and be defcribed about the triangle ABC, which
was to be done.

COR. And it is manifeft, that, when the centre of the circle
falls within the triangle, each of its angles is lefs than a right
angle, each of them being in a fegment greater than a femicir-
cle; but, when the centre is in one of the fides of the triangle,
the angle oppofite to this fide, being in a femicircle, is a right
angle; and, if the centre falls without the triangle, the angle
oppofite to the fide beyond which it is, being in a fegment lefs
than a femicircle, is greater than a right angle: Wherefore, if
the given triangle be acute angled, the centre of the circle falls
within it; if it be a right angled triangle, the centre is in the
fide oppofite to the right angle; and, if it be an obtufe angled
triangle, the centre falls without the triangle, beyond the fide
oppofite to the obtufe angle.

P R O P. VI. P R O B.

T O infcribe a fquare in a given circle.

Let ABCD be the given circle; it is required to infcribe a
fquare in ABCD.

Draw the diameters AC, BD at right angles to one an-
other; and join AB, BC, CD, DA; becaufe BE is equal to
ED, for E is the centre, and that EA
is common, and at right angles to
BD; the bafe BA is equal ᵃ to the
bafe AD; and, for the fame reafon,
BC, CD are each of them equal to
BA or AD; therefore the quadri-
lateral figure ABCD is equilateral.
It is alfo rectangular; for the ftraight
line BD, being the diameter of the
circle ABCD, BAD is a femicircle;
wherefore the angle BAD is a right

a 4. 1.

angle; for the fame reafon each of the angles ABC, BCD, ᵇ 31. 3.
CDA is a right angle; therefore the quadrilateral figure ABCD
is rectangular, and it has been fhown to be equilateral; there-
fore it is a fquare; and it is infcribed in the circle ABCD.
Which was to be done.
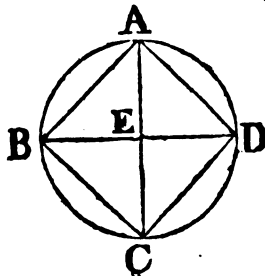
P R O P.

PROP. VII. PROB.

To deſcribe a ſquare about a given circle.

Let ABCD be the given circle ; it is required to deſcribe a ſquare about it.

Draw two diameters AC, BD of the circle ABCD, at right angles to one another, and through the points A, B, C, D draw [a] FG, GH, HK, KF touching the circle ; and becauſe FG touches the circle ABCD, and EA is drawn from the centre E to the point of contact A, the angles at A are right [b] angles ; for the ſame reaſon, the angles at the points B, C, D are right angles ; and becauſe the angle AEB is a right angle, as likewiſe is EBG, GH is pa- rallel [c] to AC; for the ſame reaſon, AC is parallel to FK, and in like manner GF, HK may each of them be demonſtrated to be parallel to BED; therefore the figures GK, GC, AK, FB, BK are parallelograms ; and GF is therefore equal [d] to HK, and GH to FK ; and becauſe AC is equal to BD, and that AC is equal to each of the two GH, FK ; and BD to each of the two GF, HK : GH FK are each of them equal to GF or HK ; therefore the quadrilateral figure FGHK is equi- lateral. It is alſo rectangular ; for GBEA being a parallelo- gram, and AEB a right angle, AGB [d] is likewiſe a right angle : In the ſame manner, it may be ſhown that the angles at H, K, F are right angles ; therefore the quadrilateral figure FGHK is rectangular, and it was demonſtrated to be equilateral ; there- fore it is a ſquare ; and it is deſcribed about the circle ABCD. Which was to be done.

a 17. 3.

b 18. 3.

c 28. 1.

d 34. 1.
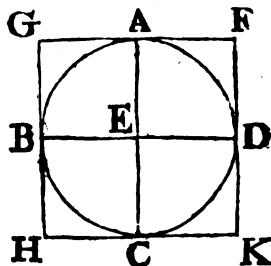
PROP. VIII. PROB.

To inſcribe a circle in a given ſquare.

Let ABCD be the given ſquare ; it is required to inſcribe a circle in ABCD.

Biſect [a] each of the ſides AB, AD, in the points F, E, and through E draw [b] EH parallel to AB or DC, and through F draw
FK

a 10. 1.

b 31. 1.

FK parallel to AD or BC; therefore each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram, and their opposite sides are equal c; and because AD is equal to AB, and that AE is the half of AD, and AF the half of AB, AE is equal to AF; wherefore the sides opposite to these are equal, viz. FG to GE; in the same manner, it may be demonstrated that GH, GK are each of them equal to FG or GE; therefore the four straight lines GE, GF, GH, GK, are equal to one another; and the circle described from the centre G, at the distance of one of them, shall pass thro' the extremities of the other three, and touch the straight lines AB, BC, CD, DA; because the angles at the points E, F, H, K are right d angles, and that the straight line which is drawn from the extremity of a diameter, at right angles to it, touches the circle e; therefore each of the straight lines AB, BC, CD, DA touches the circle, which therefore is inscribed in the square ABCD. Which was to be done.

c 34. 1.

d 29. 1.

e 16. 3.
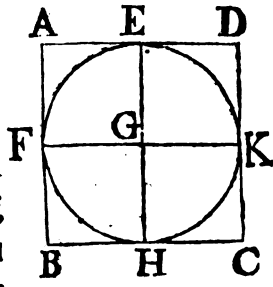
P R O P. IX.   P R O B.

T O describe a circle about a given square.

Let ABCD be the given square; it is required to describe a circle about it.

Join AC, BD cutting one another in E; and because DA is equal to AB, and AC common to the triangles DAC, BAC, the two sides DA, AC are equal to the two BA, AC; and the base DC is equal to the base BC; wherefore the angle DAC is equal a to the angle BAC, and the angle DAB is bisected by the straight line AC: In the same manner, it may be demonstrated that the angles ABC, BCD, CDA are severally bisected by the straight lines BD, AC; therefore, because the angle DAB is equal to the angle ABC, and that the angle EAB is the half of DAB, and EBA the half of ABC; the angle EAB is equal to the angle EBA; wherefore the side EA is equal b to the side EB: In the same manner, it may be b 6. 1. demonstrated

a 8. 1.
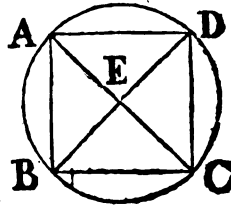
demonſtrated that the ſtraight lines EC, ED are eaeh of them equal to EA or EB; therefore the four ſtraight lines EA, EB, EC, ED are equal to one another; and the circle deſcribed from the centre E, at the diſtance of one of them, ſhall paſs through the extremities of the other three, and be deſcribed about the ſquare ABCD. Which was to be done.

PROP. X. PROB.

TO deſcribe an iſoſceles triangle, having each of the angles at the baſe double of the third angle.

a II. 2.   Take any ſtraight line AB, and divide ᵃ it in the point C, ſo that the rectangle AB, BC be equal to the ſquare of CA ; and from the centre A, at the diſtance AB, deſcribe the circle BDE,

b I. 4.   in which place ᵇ the ſtraight line BD equal to AC, whieh is not greater than the diameter of the circle BDE; join DA, DC,

c 5. 4.   and about the triangle ADC deſcribe ᶜ the circle ACD; the triangle ABD is ſuch as is required, that is, each of the angles ABD, ADB is double of the angle BAD.

     Becauſe the rectangle AB, BC is equal to the ſquare of AC, and that AC is equal to BD, the rectangle AB, BC is equal to

the ſquare of BD ; and becauſe
from the point B without the
circle ACD two ſtraight lines
BCA, BD are drawn to the cir-
cumference, one of which cuts,
and the other meets the circle,
and that the rectangle AB, BC
contained by the whole of the
cutting line, and the part of it
without the circle, is equal to the
ſquare of BD which meets it;

d 37. 3.   the ſtraight line BD touches ᵈ
the circle ACD ; and becauſe
BD touches the circle, and DC
is drawn from the point of con-

e 32. 3.   tact D, the angle BDC is equal ᵉ to the angle DAC in the alternate ſegment of the circle ; to each of theſe add the angle CDA ; therefore the whole angle BDA is equal to the two

f 32. 1.   angles CDA, DAC ; but the exterior angle BCD is equal ᶠ to the angles CDA, DAC; therefore alſo BDA is equal to BCD ;
     but
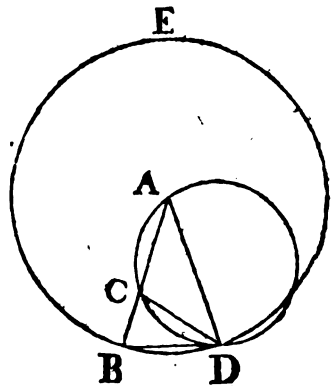
but BDA is equal ᵍ to the angle CBD, becaufe the fide AD is equal to the fide AB; therefore CBD, or DBA is equal to BCD; and confequently the three angles BDA, DBA, BCD, are equal to one another; and becaufe the angle DBC is equal to the angle BCD, the fide BD is equal ʰ to the fide DC; but BD was made equal to CA; therefore alfo CA is equal to CD, and the angle CDA equal ᵍ to the angle DAC; therefore the angles CDA, DAC together, are double of the angle DAC: But BCD is equal to the angles CDA, DAC; therefore alfo BCD is double of DAC, and BCD is equal to each of the angles BDA, DBA; each therefore of the angles BDA, DBA is double of the angle DAB; wherefore an ifofceles triangle ABD is defcribed, having each of the angles at the bafe double of the third angle. Which was to be done.

ʰ 6, 1.

PROP. XI. PROB.

TO infcribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given circle; it is required to infcribe an equilateral and equiangular pentagon in the circle ABCDE. Defcribe ᵃ an ifofceles triangle FGH, having each of the angles at G, H, double of the angle at F; and in the circle ABCDE infcribe ᵇ the triangle ACD equiangular to the triangle FGH, fo that the angle CAD be equal to the angle at F, and each of the angles ACD, CDA equal to the angle at G or H; wherefore each of the angles ACD, CDA is double of the angle CAD. Bifect ᶜ the angles ACD, CDA by the ftraight lines CE, DB; and join AB, BC, DE, EA. ABCDE is the pentagon required.

ᵃ 10. 4.

ᵇ 2. 4.

ᶜ 9. 1.
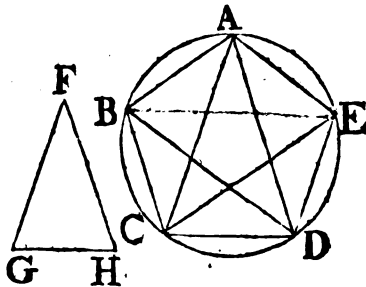
Becaufe each of the angles ACD, CDA is double of CAD, and are bifected by the ftraight lines CE, DB, the five angles DAC, ACE, ECD, CDB, BDA are equal to one another; but equal angles ftand upon equal circumferences; therefore the five circumferences AB, BC, CD, DE, EA are equal to one another:

ᵈ 26. 3.

e 29. 3.

another : And equal circumferences are fubtended by equal ftraight lines ; therefore the five ftraight lines AB, BC, CD, DE, EA are equal to one another.   Wherefore the pentagon ABCDE is equilateral. It is alfo equiangular; becaufe the circumference AB is equal to the circumference DE : If to each be added BCD, the whole ABCD is equal to the whole EDCB : And the angle AED ftands on the circumference ABCD, and the angle BAE on the circumference EDCB ; therefore the angle BAE is equal f to the angle AED : For the fame reafon, each of the angles ABC, BCD, CDE is equal to the angle BAE, or AED : Therefore the pentagon ABCDE is equiangular ; and it has been fhown that it is equilateral. Wherefore, in the given circle, an equilateral and equiangular pentagon has been inf cribed.   Which was to be done.

f 27. 3.

P R O P. XII. P R O B.

TO defcribe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be the given circle ; it is required to defcribe an equilateral and equiangular pentagon about the circle ABCDE.

a 11. 4.

b 17. 3.

c 18. 3.

d 47. 1.

Let the angles of a pentagon, infcribed in the circle, by the laft propofition, be in the points A, B, C, D, E, fo that the circumferences AB, BC, CD, DE, EA are equal a ; and thro' the points A, B, C, D, E draw GH, HK, KL, LM, MG, touching b the circle ; take the centre F, and join FB, FK, FC, FL, FD : And becaufe the ftraight line KL touches the circle ABCDE in the point C, to which FC is drawn from the centre F, FC is perpendicular c to KL ; therefore each of the angles at C is a right angle : For the fame reafon, the angles at the points B, D are right angles : And becaufe FCK is a right angle, the fquare of FK is equal d to the fquares of FC, CK : For the fame reafon, the fquare of FK is equal to the fquares of FB, BK : Therefore the fquares of FC, CK are equal to the fquares of FB, BK, of which the fquare of FC is equal to the fquare of FB ; the remaining fquare of CK is therefore equal to
the

the remaining fquare of BK, and the ftraight line CK equal to BK: And becaufe FB is equal to FC, and FK common to the triangles BFK, CFK, the two BF, FK are equal to the two CF, FK; and the bafe BK is equal to the bafe KC; therefore the angle BFK is equal ᵉ to the angle KFC, and the angle BKF to FKC; wherefore the angle BFC is double of the angle KFC, and BKC double of FKC: For the fame reafon, the angle CFD is double of the angle CFL, and CLD double of CLF: And be-caufe the circumference BC is equal to the circumference CD,

e 8. 1.

the angle BFC is equal ᶠ to the angle CFD; and BFC is dou-ble of the angle KFC, and CFD double of CFL; there fore the angle KFC is equal to the angle CFL; and the right angle FCK is equal to the right angle FCL: Therefore, in the two triangles FKC, FLC, there are two angles of one equal to two angles of the other, each to each, and the fide FC, which

f 27. 3.

is adjacent to the equal angles in each, is common to both; therefore the other fides fhall be equal ᵍ to the other fides, and the third angle to the third angle: Therefore the ftraight line KC is equal to CL, and the angle FKC to the angle FLC: And becaufe KC is equal to CL, KL is double of KC: In the fame manner, it may be fhown that HK is double of BK: And becaufe BK is equal to KC, as was demonftrated, and that KL is double of KC, and HK double of BK, HK fhall be equal to KL: In like manner, it may be fhown that GH, GM, ML are each of them equal to HK or KL: Therefore the pentagon GHKLM is equilateral. It is alfo equiangular; for, fince the angle FKC is equal to the angle FLC, and that the angle HKL is double of the angle FKC, and KLM double of FLC, as was before demonftrated, the angle HKL is equal to KLM: And in like manner it may be fhown, that each of the angles KHG, HGM, GML is equal to the angle HKL or KLM: There-fore the five angles GHK, HKL, KLM, LMG, MGH being equal to one another, the pentagon GHKLM is equiangular: And it is equilateral, as was demonftrated; and it is defcribed about the circle ABCDE. Which was to be done.

g 26. 1.

H

PROP.

PROP. XIII. PROB.

TO infcribe a circle in a given equilateral and equi-
angular pentagon.

Let ABCDE be the given equilateral and equiangular penta-
gon; it is required to infcribe a circle in the pentagon ABCDE.

Bifect ᵃ the angles BCD, CDE by the ftraight lines CF, DF,
and from the point F, in which they meet, draw the ftraight lines
FB, FA, FE: Therefore, fince BC is equal to CD, and CF com-
mon to the triangles BCF, DCF, the two fides BC, CF are equal
to the two DC, CF; and the angle BCF is equal to the angle
DCF; therefore the bafe BF is equal ᵇ to the bafe FD, and the
other angles to the other angles, to which the equal fides are op-
pofite; therefore the angle CBF is equal to the angle CDF: And
becaufe the angle CDE is double of CDF, and that CDE is equal
to CBA, and CDF to CBF; CBA
is alfo double of the angle CBF;
therefore the angle ABF is equal
to the angle CBF; wherefore the
angle ABC is bifected by the
ftraight line BF: In the fame
manner, it may be demonftrated,
that the angles BAE, AED are
bifected by the ftraight lines AF,
FE: From the point F draw ᶜ
FG, FH, FK, FL, FM perpen-
diculars to the ftraight lines AB,
BC, CD, DE, EA: And be-
caufe the angle HCF is equal to
KCF, and the right angle FHC equal to the right angle FKC;
in the triangles FHC, FKC there are two angles of one equal
to two angles of the other, and the fide FC, which is oppofite
to one of the equal angles in each, is common to both; therefore
the other fides fhall be equal ᵈ, each to each; wherefore the
perpendicular FH is equal to the perpendicular FK: In the fame
manner it may be demonftrated that FL, FM, FG are each of
them equal to FH or FK; therefore the five ftraight lines FG,
FH, FK, FL, FM are equal to one another: Wherefore the cir-
cle defcribed from the centre F, at the diftance of one of thefe
five, fhall pafs through the extremities of the other four, and
touch

touch the ſtraight lines AB, BC, CD, DE, EA, becauſe the
angles at the points G, H, K, L, M are right angles; and that
a ſtraight line drawn from the extremity of the diameter of a
circle at right angles to it, touches e the circle: Therefore each e 16. 3.
of the ſtraight lines AB, BC, CD, DE, EA touches the circle;
wherefore it is inſcribed in the pentagon ABCDE. Which
was to be done.

PROP. XIV. PROB.

TO deſcribe a circle about a given equilateral and
equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pen-
tagon; it is required to deſcribe a circle about it.
Biſect a the angles BCD, CDE by the ſtraight lines CF, FD, a 9. 1.
and from the point F, in which they meet, draw the ſtraight
lines FB, FA, FE to the points B,
A, E. It may be demonſtrated, in
the ſame manner as in the preceding
propoſition, that the angles CBA,
BAE, AED are biſected by the
ſtraight lines FB, FA, FE: And
becauſe the angle BCD is equal to
the angle CDE, and that FCD is
the half of the angle BCD, and CDF
the half of CDE; the angle FCD is
equal to FDC; wherefore the ſide
CF is equal b to the ſide FD: In like manner it may be demon- b 6. 1.
ſtrated that FB, FA, FE are each of them equal to FC or FD:
Therefore the five ſtraight lines FA, FB, FC, FD, FE are
equal to one another; and the circle deſcribed from the centre
F, at the diſtance of one of them, ſhall paſs through the extre-
mities of the other four, and be deſcribed about the equilateral
and equiangular pentagon ABCDE. Which was to be done.
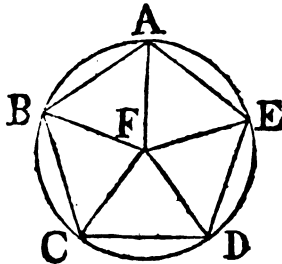
PROP.

P R O P. XV. P R O B.

See N.

TO inſcribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given circle; it is·required to inſcribe an equilateral and equiangular hexagon in it.

Find the centre G of the circle ABCDEF, and draw the diameter AGD; and from D as a centre, at the diſtance DG, deſcribe the circle EGCH, join EG, CG, and produce them to the points B, F; and join AB, BC, CD, DE, EF, FA; The hexagon ABCDEF is equilateral and equiangular.

Becauſe G is the centre of the circle ABCDEF, GE is equal to GD: And becauſe D is the centre of the circle EGCH, DE is equal to DG; wherefore. GE is equal to ED, and the triangle EGD is equilateral; and therefore its three angles EGD, GDE, DEG are equal to one another, becauſe the angles at

a 5. 1.
b 32. 1.

the baſe of an iſoſceles triangle are equal [a]; and the three angles of a triangle are equal [b] to two right angles; therefore the angle EGD is the third part of two right angles: In the ſame

manner it may be demonſtrated that the angle DGC is alſo the third part of two right angles: And becauſe the ſtraight line GC makes with EB the

c 13. 1.

adjacent angles EGC, CGB equal [c] to two right angles; the remaining angle CGB is the third part of two right angles; therefore the angles EGD, DGC, CGB are equal to one

d 15. 1.

another: And to theſe are equal [d] the vertical oppoſite angles BGA, AGF, FGE: Therefore the ſix angles EGD, DGC, CGB, BGA, AGF, FGE are equal to one another: But equal

e 26. 3.

angles ſtand upon equal [e] circumferences; therefore the ſix circumfe

f 9. 3.

rences AB, BC, CD, DE, EF, FA are equal to one another: And equal circumferences are ſubtended by equal [f] ſtraight lines; therefore the ſix ſtraight lines are equal to one another, and the hexagon ABCDEF is equilateral. It is alſo equiangular; for, ſince the circumference AF is equal to ED, to each of theſe add the circumference ABCD; therefore the whole circumference FABCD ſhall be equal to the whole EDCBA: And
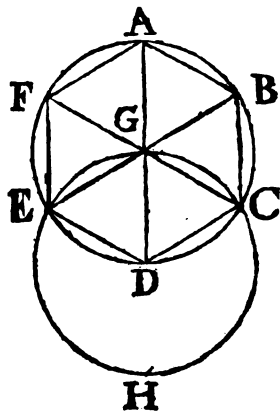
And the angle FED ſtands upon the circumference FABCD, and the angle AFE upon EDCBA; therefore the angle AFE is equal to FED: In the ſame manner it may be demonſtrated that the other angles of the hexagon ABCDEF are each of them equal to the angle AFE or FED: Therefore the hexagon is equiangular; and it is equilateral, as was ſhown; and it is inſcribed in the given circle ABCDEF.   Which was to be done.

Cor. From this it is manifeſt, that the ſide of the hexagon is equal to the ſtraight line from the centre, that is, to the ſemidiameter of the circle.

And if thro' the points A, B, C, D, E, F there be drawn ſtraight lines touching the circle, an equilateral and equiangular hexagon ſhall be deſcribed about it, which may be demonſtrated from what has been ſaid of the pentagon; and likewiſe a circle may be inſcribed in a given equilateral and equiangular hexagon, and circumſcribed about it, by a method like to that uſed for the pentagon.

PROP. XVI. PROB.

TO inſcribe an equilateral and equiangular quindeca-gon in a given circle.   See N.

Let ABCD be the given circle; it is required to inſcribe an equilateral and equiangular quindecagon in the circle ABCD.

Let AC be the ſide of an equilateral triangle inſcribed [a] in the circle, and AB the ſide of an equilateral and equiangular pentagon inſcribed [b] in the ſame; therefore, of ſuch equal parts as the whole circumference ABCDF contains fifteen, the circumference ABC, being the third part of the whole, contains five; and the circumference AB, which is the fifth part of the whole, contains three; therefore BC their difference contains two of the ſame parts: Biſect [c] BC in E; therefore BE, EC are, each of them, the fifteenth part of the whole circumference ABCD: Therefore, if the ſtraight lines BE, EC be drawn, and ſtraight lines equal to them be placed [d] around in the whole circle, an equilateral and equiangular quindecagon ſhall be inſcribed in it.   Which was to be done.

a 2. 4.

b 11. 4.

c 10. 3.

d 1. 4.
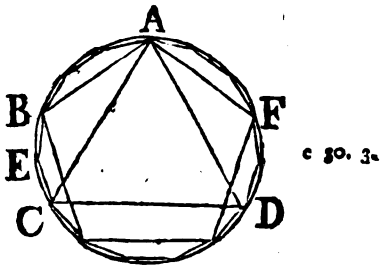
H 3   And

Book IV.
And in the same manner as was done in the pentagon, if, through the points of division made by inscribing the quindecagon, straight lines be drawn touching the circle, an equilateral and equiangular quindecagon shall be described about it: And likewise, as in the pentagon, a circle may be inscribed in a given equilateral and equiangular quindecagon, and circumscribed about it.

THE

THE

ELEMENTS

O F

EUCLID.

BOOK V.

DEFINITIONS.

I.

A Lefs magnitude is faid to be a part of a greater magnitude, when the lefs meafures the greater, that is, ' when the lefs is contained a certain number of times exactly ' in the greater.'

II.

A greater magnitude is faid to be a multiple of a lefs, when the greater is meafured by the lefs, that is, ' when the greater ' contains the lefs a certain number of times exactly.'

III.

' Ratio is a mutual relation of two magnitudes of the fame See N. ' kind to one another, in refpect of quantity.'

IV.

Magnitudes are faid to have a ratio to one another, when the lefs can be multiplied fo as to exceed the other.

V.

The firft of four magnitudes is faid to have the fame ratio to the fecond, which the third has to the fourth, when any equimultiples whatfoever of the firft and third being taken, and any equimultiples whatfoever of the fecond and fourth ; if the multiple of the firft be lefs than that of the fecond, the multiple of the third is alfo lefs than that of the fourth ; or, if the multiple of the firft be equal to that of the fecond, the multiple of the third is alfo equal to that of the fourth ;

or,

or, if the multiple of the firft be greater than that of the
fecond, the multiple of the third is alfo greater than that of
the fourth.

VI.

Magnitudes which have the fame ratio are called proportionals,
N. B. ' When four magnitudes are proportionals, it is
' ufually expreffed by faying, the firft is to the fecond, as the
' third to the fourth.'

VII.

When of the equimultiples of four magnitudes (taken as in
the fifth definition) the multiple of the firft is greater than
that of the fecond, but the multiple of the third is not
greater than the multiple of the fourth ; then the firft is faid
to have to the fecond a greater ratio than the third magni-
tude has to the fourth; and, on the contrary, the third is
faid to have to the fourth a lefs ratio than the firft has to the
fecond.

VIII.

" Analogy, or proportion, is the fimilitude of ratios."

IX.

Proportion confifts in three terms at leaft.

X.

When three magnitudes are proportionals, the firft is faid to
have to the third the duplicate ratio of that which it has to
the fecond.

XI.

When four magnitudes are continual proportionals, the firft is
faid to have to the fourth the triplicate ratio of that which it
has to the fecond, and fo on, quadruplicate, &c. increafing
the denomination ftill by unity, in any number of propor-
tionals.

Definition A, to wit, of compound ratio.

When there are any number of magnitudes of the fame kind,
the firft is faid to have to the laft of them the ratio com-
pounded of the ratio which the firft has to the fecond, and of
the ratio which the fecond has to the third, and of the ratio
which the third has to the fourth, and fo on unto the laft
magnitude.

For example, If A, B, C, D be four magnitudes of the fame
kind, the firft A is faid to have to the laft D the ratio com-
pounded of the ratio of A to B, and of the ratio of B to C,
and of the ratio of C to D ; or, the ratio of A to D is faid to
be compounded of the ratios of A to B, B to C, and C to D :

And

And if A has to B, the fame ratio which E has to F; and B to C, the fame ratio that G has to H; and C to D, the fame that K has to L; then, by this definition, A is faid to have to D the ratio compounded of ratios which are the fame with the ratios of E to F, G to H, and K to L: And the fame thing is to be underftood when it is more briefly expreffed, by faying A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the fame things being fuppofed, if M has to N the fame ratio which A has to D; then, for fhortnefs fake, M is faid to have to N, the ratio compounded of the ratios of E to F, G to H, and K to L.

XII.

In proportionals, the antecedent terms are called homologous to one another, as alfo the confequents to one another.

' Geometers make ufe of the following technical words to fig-
' nify certain ways of changing either the order or magni-
' tude of proportionals, fo as that they continue ftill to be
' proportionals.'

XIII.

Permutando, or alternando, by permutation, or alternately; See N, this word is ufed when there are four proportionals, and it is inferred, that the firft has the fame ratio to the third, which the fecond has to the fourth; or that the firft is to the third, as the fecond to the fourth: As is fhown in the 16th prop. of this 5th book.

XIV.

Invertendo, by inverfion: When there are four proportionals, and it is inferred, that the fecond is to the firft, as the fourth to the third. Prop. B. book. 5.

XV.

Componendo, by compofition; when there are four proportion-als, and it is inferred, that the firft, together with the fe-cond, is to the fecond, as the third, together with the fourth, is to the fourth. 18th prop. book 5.

XVI.

Dividendo, by divifion; when there are four proportionals, and it is inferred, that the excefs of the firft above the fecond, is to the fecond, as the excefs of the third above the fourth, is to the fourth. 17th prop. book 5.

XVII.

Convertendo, by converfion; when there are four proportion-als, and it is inferred, that the firft is to its excefs above the

fecond, as the third to its excefs above the fourth. Prop. E. book 5.

XVIII.

Ex aequali (fc. diftantia), or ex aequo, from equality of di-ftance; when there is any number of magnitudes more than two, and as many others, fo that they are proportionals when taken two and two of each rank, and it is inferred, that the firft is to the laft of the firft rank of magnitudes, as the firft is to the laft of the others: ‘ Of this there are the two ‘ following kinds, which arife from the different order in ‘ which the magnitudes are taken two and two.’

XIX.

Ex aequali, from equality; this term is ufed fimply by itfelf, when the firft magnitude is to the fecond of the firft rank, as the firft to the fecond of the other rank; and as the fe-cond is to the third of the firft rank, fo is the fecond to the third of the other; and fo on in order, and the inference is as mentioned in the preceding definition; whence this is called ordinate proportion. It is demonftrated in 22d prop. book 5.

XX.

Ex aequali, in proportione perturbata, feu inordinata; from e-quality, in perturbate or diforderly proportion *; this term is ufed when the firft magnitude is to the fecond of the firft rank, as the laft but one is to the laft of the fecond rank; and as the fecond is to the third of the firft rank, fo is the laft but two to the laft but one of the fecond rank; and as the third is to the fourth of the firft rank, fo is the third from the laft to the laft but two of the fecond rank; and fo on in a crofs order: And the inference is as in the 18th definition. It is demonftrated in the 23d prop. of book 5.

A X I O M S.

I.

EQuimultiples of the fame, or of equal magni-tudes, are equal to one another.

II. Thofe

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* 4. Prop. lib. 2. Archimedis de fphaera et cylindro.

II.

Thofe magnitudes of which the fame, or equal magnitudes, are equimultiples, are equal to one another.

III.

A multiple of a greater magnitude is greater than the fame multiple of a lefs.

IV.

That magnitude of which a multiple is greater than the fame multiple of another, is greater than that other magnitude.

PROP. I. THEOR.

IF any number of magnitudes be equimultiples of as many, each of each; what multiple foever any one of them is of its part, the fame multiple fhall all the firft magnitudes be of all the other.

Let any number of magnitudes AB, CD be equimultiples of as many others E, F, each of each; whatfoever multiple AB is of E, the fame multiple fhall AB and CD together be of E and F together.

Becaufe AB is the fame multiple of E that CD is of F, as many magnitudes as are in AB equal to E, fo many are there in CD equal to F. Divide AB into magnitudes equal to E, viz. AG, GB; and CD into CH, HD equal each of them to F: The number therefore of the magnitudes CH, HD fhall be equal to the number of the others AG, GB: And becaufe AG is equal to E, and CH to F, therefore AG and CH together are equal to ᵃ E and F together: For the fame reafon, becaufe GB is equal to E, and HD to F; GB and HD together are equal to E and F together. Wherefore, as many magnitudes as are in AB equal to E, fo many are there in AB, CD together equal to E and F together. Therefore, whatfoever multiple AB is of E, the fame multiple is AB and CD together of E and F together.
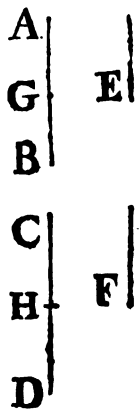
ᵃ Ax. 2. 1.

Therefore, if any magnitudes, how many foever, be equimultiples of as many, each of each, whatfoever multiple any one of them is of its part, the fame multiple fhall all the firft magnitudes be of all the other: ' For the fame demonftration ' holds

' holds in any number of magnitudes, which was here applied
' to two.' Q. E. D.

PROP. II. THEOR

IF the firſt magnitude be the ſame multiple of the ſe-
cond that the third is of the fourth, and the fifth the
ſame multiple of the ſecond that the ſixth is of the
fourth ; then ſhall the firſt together with the fifth be the
ſame multiple of the ſecond, that the third together with
the ſixth is of the fourth.

Let AB the firſt, be the ſame multiple of C the ſecond, that
DE the third is of F the fourth ; and BG the fifth, the ſame
multiple of C the ſecond, that EH
the ſixth is of F the fourth : Then
is AG the firſt, together with the fifth,
the ſame multiple of C the ſecond,
that DH the third, together with the
ſixth, is of F the fourth.
Becauſe AB is the ſame multiple
of C, that DE is of F ; there are as
many magnitudes in AB equal to C,
as there are in DE equal to F : In like
manner, as many as there are in BG
equal to C, ſo many are there in EH equal to F : As many,
then, as are in the whole AG equal to C, ſo many are there in
the whole DH equal to F : Therefore AG is the ſame multi-
ple of C, that DH is of F ; that is, AG the firſt and fifth to-
gether, is the ſame multiple of the ſecond
C, that DH the third and ſixth together is
of the fourth F. If, therefore, the firſt be
the ſame multiple, &c. Q. E. D.

Cor. ' From this it is plain, that, if any
' number of magnitudes AB, BG, GH,
' be multiples of another C ; and as many
' DE, EK, KL be the ſame multiples of
' F, each of each ; the whole of the firſt,
' viz. AH, is the ſame multiple of C,
' that the whole of the laſt, viz. DL, is
' of F.'
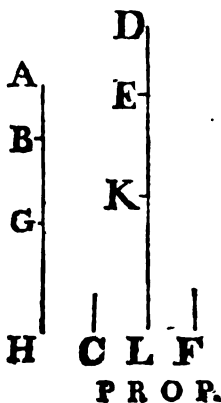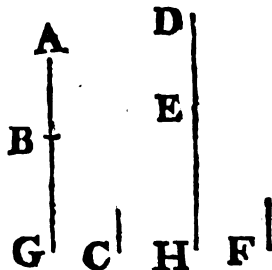
PROP.

PROP. III. THEOR.

IF the first be the same multiple of the second, which the third is of the fourth ; and if of the first and third there be taken equimultiples, these shall be equimultiples, the one of the second, and the other of the fourth.

Let A the first, be the same multiple of B the second, that C the third is of D the fourth ; and of A, C let the equimultiples EF, GH be taken : Then EF is the same multiple of B, that GH is of D.

Because EF is the same multiple of A, that GH is of C, there are as many magnitudes in EF equal to A, as are in GH equal to C : Let EF be divided into the magnitudes EK, KF, each equal to A, and GH into GL, LH, each equal to C : The number therefore of the magnitudes EK, KF, shall be equal to the number of the others GL, LH : And because A is the same multiple of B, that C is of D, and that EK is equal to A, and GL to C ; therefore EK is the same multiple of B, that GL is of D : For the same reason, KF is the same multiple of B, that LH is of D ; and so, if there be more parts in EF, GH equal to A, C : Because, therefore, the first EK is the same multiple of the second B, which the third GL is of the fourth D, and that the fifth KF is the same multiple of the second B, which the sixth LH is of the fourth D ; EF the first, together with the fifth, is the same multiple * of the second B, which GH the third, together with the sixth, is of the fourth D. If, therefore, the first, &c. Q. E. D.
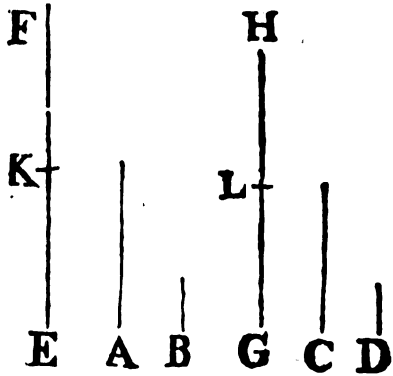
* 2. 5²

PROP.

PROP. IV. THEOR.

See N. IF the firſt of four magnitudes has the ſame ratio to
the ſecond which the third has to the fourth; then
any equimultiples whatever of the firſt and third ſhall
have the ſame ratio to any equimultiples of the ſecond
and fourth, viz. ' the equimultiple of the firſt ſhall have
' the ſame ratio to that of the ſecond, which the equi-
' multiple of the third has to that of the fourth.'

Let A the firſt, have to B the ſecond, the ſame ratio which
the third C has to the fourth D; and of A and C let there be
taken any equimultiples whatever
E, F; and of B and D any equi-
multiples whatever G, H : Then
E has the ſame ratio to G, which
F has to H.

Take of E and F any equimul-
tiples whatever K, L, and of G,
H, any equimultiples whatever M,
N : Then, becauſe E is the ſame
multiple of A, that F is of C;
and of E and F have been taken
equimultiples K, L; therefore K
is the ſame multiple of A, that L

a 3. 5. is of C a : For the ſame reaſon, M
is the ſame multiple of B, that N
is of D : And becauſe, as A is to

b Hypoth. B, ſo is C to D b, and of A and
C have been taken certain equi-
multiples K, L; and of B and D
have been taken certain equimul-
tiples M, N; if therefore K be
greater than M, L is greater than
N ; and if equal, equal; if leſs,

c 5. def. 5. leſs c. And K, L are any equi-
multiples whatever of E, F; and
M, N any whatever of G, H:
As therefore E is to G, ſo is c F
to H. Therefore, if the firſt, &c.
Q. E. D.

$$\begin{array}{cccccc} K & E & A & B & G & M \\ L & F & C & D & H & N \end{array}$$

Cor. Likewiſe, if the firſt has the ſame ratio to the ſecond,
which the third has to the fourth, then alſo any equimulti-
ples
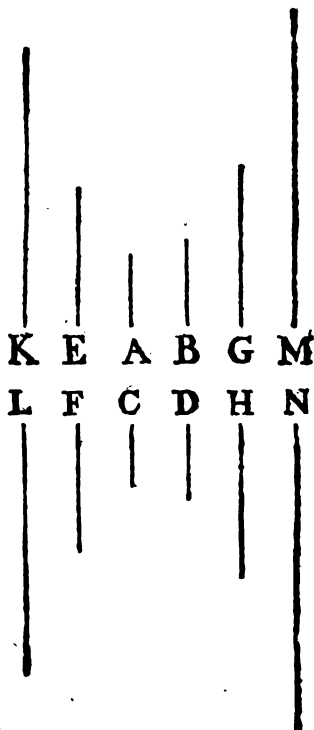
ples whatever of the firſt and third have the ſame ratio to the
ſecond and fourth : And in like manner, the firſt and the third
have the ſame ratio to any equimultiples whatever of the ſecond
and fourth.

Let A the firſt, have to B the ſecond, the ſame ratio which
the third C has to the fourth D, and of A and C let E and F
be any equimultiples whatever ; then E is to B, as F to D.

Take of E, F any equimultiples whatever K, L, and of B,
D any equimultiples whatever G, H ; then it may be demon-
ſtrated, as before, that K is the ſame multiple of A, that L is
of C : And becauſe A is to B, as C is to D, and of A and C
certain equimultiples have been taken, viz. K and L ; and of
B and D certain equimultiples G, H ; therefore, if K be greater
than G, L is greater than H ; and if equal, equal; if leſs, leſs ᶜ : c 5. def. 5.
And, K, L are any equimultiples of E, F, and G, H any what-
ever of B, D ; as therefore E is to B, ſo is F to D : And in the
ſame way the other caſe is demonſtrated.

PROP. V. THEOR.

IF one magnitude be the ſame multiple of another, See N.
which a magnitude taken from the firſt is of a mag-
nitude taken from the other ; the remainder ſhall be the
ſame multiple of the remainder, that the whole is of the
whole.

Let the magnitude AB be the ſame multiple
of CD, that AE taken from the firſt, is of CF
taken from the other ; the remainder EB ſhall
be the ſame multiple of the remainder FD, that
the whole AB is of the whole CD.

Take AG the ſame multiple of FD, that
AE is of CF : Therefore AE is ᵃ the ſame mul-   a 1. 5.
tiple of CF, that EG is of CD : But AE, by
the hypotheſis, is the ſame multiple of CF, that
AB is of CD : Therefore EG is the ſame mul-
tiple of CD that AB is of CD ; wherefore EG
is equal to AB ᵇ. Take from them the common   b 1. Ax. 5.
magnitude AE ; the remainder AG is equal to
the remainder EB. Wherefore, ſince AE is
the ſame multiple of CF, that AG is of FD,
and that AG is equal to EB ; therefore AE is the ſame multiple
of CF, that EB is of FD : But AE is the ſame multiple of CF,
that
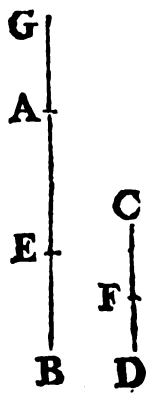
that AB is of CD ; therefore EB is the fame multiple of FD,
that AB is of CD. Therefore, if any magnitude, &c. Q. E. D.

PROP. VI. THEOR.

See N.   IF two magnitudes be equimultiples of two others, and
if equimultiples of thefe be taken from the firft two,
the remainders are either equal to thefe others, or equi-
multiples of them.

Let the two magnitudes AB, CD be equimultiples of the two
E, F, and AG, CH taken from the firft two be equimultiples
of the fame E, F; the remainders GB, HD are either equal
to E, F, or equimultiples of them.

Firft, Let GB be equal to E ; HD is e-
qual to F : Make CK equal to F ; and be-
caufe AG is the fame multiple of E, that
CH is of F, and that GB is equal to E,
and CK to F; therefore AB is the fame
multiple of E, that KH is of F.   But AB,
by the hypothefis, is the fame multiple of
E that CD is of F; therefore KH is the
fame multiple of F, that CD is of F;
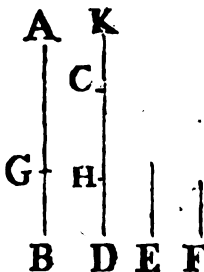
a 1. Ax. 5. wherefore KH is equal to CD [a] : Take a-
way the common magnitude CH, then the
remainder KC is equal to the remainder
HD : But KC is equal to F ; HD therefore is equal to F.

But let GB be a multiple of E ; then
HD is the fame multiple of F: Make
CK the fame multiple of F, that GB is
of E : And becaufe AG is the fame mul-
tiple of E, that CH is of F ; and GB the
fame multiple of E, that CK is of F ;
therefore AB is the fame multiple of E,
b 2. 5. that KH is of F [b] : But AB is the fame
multiple of E, that CD is of F; therefore
KH is the fame multiple of F, that CD is
of it ; wherefore KH is equal to CD [a] :
Take away CH from both ; therefore the
remainder KC is equal to the remainder
HD : And becaufe GB is the fame multiple of E, that KC is
of F, and that KC is equal to HD ; therefore HD is the fame
multiple of F, that GB is of E : If therefore two magnitudes,
&c. Q. E. D.
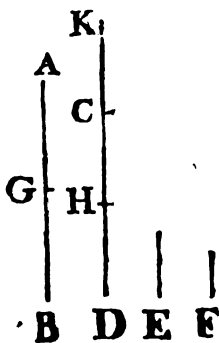
PROP.

PROP. A.   THEOR.

IF the firft of four magnitudes has to the fecond, the See N.
fame ratio which the third has to the fourth; then, if
the firft be greater than the fecond, the third is alfo
greater than the fourth; and, if equal, equal; if lefs, lefs.

Take any equimultiples of each of them, as the doubles of
each; then, by def. 5th of this book, if the double of the firft be
greater than the double of the fecond, the double of the third is
greater than the double of the fourth; but, if the firft be greater
than the fecond, the double of the firft is greater than the double
of the fecond; wherefore alfo the double of the third is greater
than the double of the fourth; therefore the third is greater than
the fourth: In like manner, if the firft be equal to the fecond,
or lefs than it, the third can be proved to be equal to the fourth,
or lefs than it.   Therefore, if the firft, &c. Q. E. D.

PROP. B.   THEOR.

IF four magnitudes are proportionals, they are propor- See N.
tionals alfo when taken inverfely.

If the magnitude A be to B, as C is to D, then alfo inverfely
B is to A, as D to C.
Take of B and D any equimultiples
whatever E and F; and of A and C any e-
quimultiples whatever G and H. Firft, Let
E be greater than G, then G is lefs than E;
and, becaufe A is to B, as C is to D, and
of A and C, the firft and third, G and H
are equimultiples; and of B and D, the fe-
cond and fourth, E and F are equimulti-
ples; and that G is lefs than E, H is alfo
ᵃ lefs than F; that is, F is greater than H;
if therefore E be greater than G, F is great-
er than H: In like manner, if E be equal
to G, F may be fhown to be equal to H;
and, if lefs, lefs; and E, F are any equi-
multiples whatever of B and D, and G, H
any whatever of A and C; therefore, as B

G A B E
H C D F   ᵃ 5. def. 5.

I
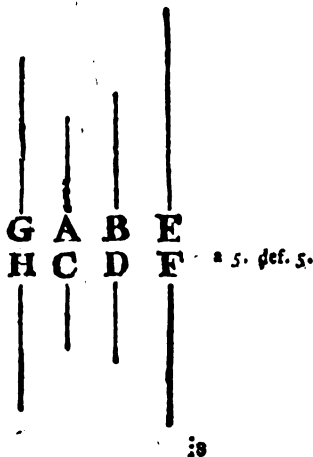
is to A, fo is D to C.   If, then, four magnitudes, &c.
Q. E. D.

PROP. C.   THEOR.

See N.

IF the firft be the fame multiple of the fecond, or the
fame part of it, that the third is of the fourth ; the
firft is to the fecond, as the third is to the fourth.

Let the firft A, be the fame multiple of B
the fecond, that C the third is of the fourth
D : A is to B as C is to D.
Take of A and C any equimultiples what-
ever E and F ; and of B and D any equi-
multiples whatever G and H : Then, becaufe
A is the fame multiple of B that C is of D ;
and that E is the fame multiple of A, that
F is of C ; E is the fame multiple of B, that
a 3. 5.   F is of D a ; therefore E and F are the fame
multiples of B and D : But G and H are equi-
multiples of B and D ; therefore, if E be a
greater multiple of B, than G is, F is a great-
er multiple of D, than H is of D; that is,
if E be greater than G, F is greater than H :
In like manner, if E be equal to G, or lefs ;
F is equal to H, or lefs than it.   But E, F
are equimultiples, any whatever, of A, C,
and G, H any equimultiples whatever of B,
b 5. def. 5. D.   Therefore A is to B, as C is to D b.

Next, Let the firft A be the fame part
of the fecond B, that the third C is of
the fourth D : A is to B, as C is to D :
For B is the fame multiple of A, that D
is of C ; wherefore, by the preceding
cafe, B is to A, as D is to C ; and in-
c B. 5.   verfely c A is to B, as C is to D. There-
fore, if the firft be the fame multiple, &c.
Q. E. D.

A B C D
E G F H

A   B   C   D
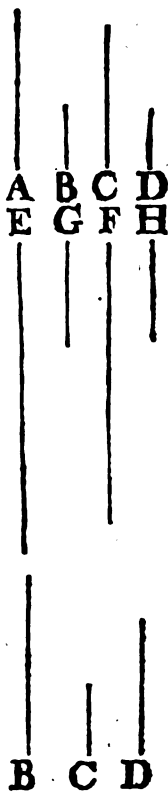
PROP.

P R O P. D. T H E O R.

IF the firft be to the fecond as the third to the fourth, See N.
and if the firft be a multiple, or part of the fecond;
the third is the fame multiple, or the fame part of the
fourth.

Let A be to B, as C is to D; and firft let A be a multiple
of B; C is the fame multiple of D.
Take E equal to A, and whatever mul-
tiple A or E is of B, make F the fame mul-
tiple of D: Then, becaufe A is to B, as C is
to D; and of B the fecond, and D the fourth
equimultiples have been taken E and F;
A is to E, as C to F^a: But A is equal
to E, therefore C is equal to F^b: And F
is the fame multiple of D, that A is of B.
Wherefore C is the fame multiple of D,
that A is of B.
Next, Let the firft A be a part of the fe-
cond B; C the third is the fame part of the
fourth D.
Becaufe A is to B, as C is to D; then,
inverfely, B is^c to A, as D to C: But A is
a part of B, therefore B is a multiple of A;
and, by the preceding cafe, D is the fame
multiple of C, that is, C is the fame part of D, that A is of B:
Therefore, if the firft, &c. Q. E. D.
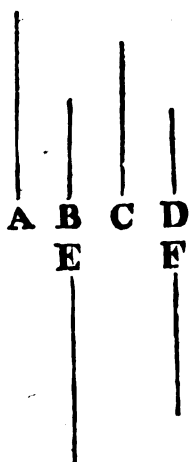
a Cor. 4. 5.

b A. 5.

See the fi-
gure at the
foot of the
preceding
page.

c B. 5.

A   B   C   D
    E   F

P R O P. VII. T H E O R.

EQUAL magnitudes have the fame ratio to the fame
magnitude; and the fame has the fame ratio to e-
qual magnitudes.

Let A and B be equal magnitudes, and C any other. A and
B have each of them the fame ratio to C, and C has the fame
ratio to each of the magnitudes A and B.
Take of A and B any equimultiples whatever D and E, and

of

of C any multiple whatever F: Then, because D is the same multiple of A, that E is of B, and that A is equal to B; D is ^{a} equal to E: Therefore, if D be greater than F, E is greater than F; and if equal, equal; if less, less: And D, E are any equimultiples of A, B, and F is any multiple of C. Therefore ^{b}, as A is to C, so is B to C.

Likewise C has the same ratio to A, that it has to B: For, having made the same construction, D may in like manner be shown equal to E: Therefore, if F be greater than D, it is likewise greater than E; and if equal, equal, if less, less: And F is any multiple whatever of C, and D, E are any equimultiples whatever of A, B. Therefore C is to A, as C is to B ^{b}. Therefore equal magnitudes, &c. Q. E. D.
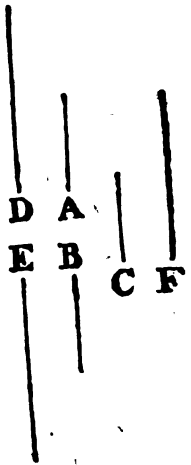
P R O P. VIII.   T H E O R.

OF unequal magnitudes, the greater has a greater ratio to the same than the less has; and the same magnitude has a greater ratio to the less, than it has to the greater.

Let AB, BC be unequal magnitudes, of which AB is the greater, and let D be any magnitude whatever: AB has a greater ratio to D than BC to D: And D has a greater ratio to BC than unto AB.

If the magnitude which is not the greater of the two AC, CB, be not less than D, take EF, FG, the doubles of AC, CB, as in Fig. 1. But, if that which is not the greater of the two AC, CB be less than D (as in Fig. 2. and 3.) this magnitude can be multiplied, so as to become greater than D, whether it be AC, or CB. Let it be multiplied, until it become greater than D, and let the other be multiplied as often; and let EF be the multiple thus taken of AC, and FG the same multiple of CB: Therefore EF and FG are each of them greater than

Fig. 1.
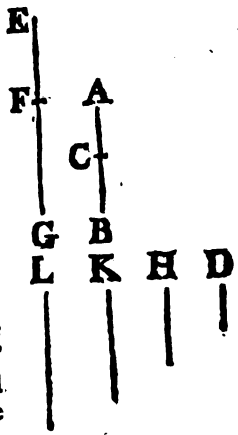
D:

D : And in every one of the cafes, take H the double of D, K
its triple, and fo on, till the multiple of D be that which firft
becomes greater than FG : Let L be that multiple of D which
is firft greater than FG, and K the multiple of D which is
next lefs than L.

Then, becaufe L is the multiple of D, which is the firft that
becomes greater than FG, the next preceding multiple K is
not greater than FG; that is, FG is not lefs than K : And fince
EF is the fame multiple of AC, that FG is of CB; FG is the
fame multiple of CB, that EG is of AB a; wherefore EG and   a 1. 5.
FG are equimultiples of AB and CB : And it was fhown, that
FG was not lefs than K,   Fig. 2.   Fig. 3.
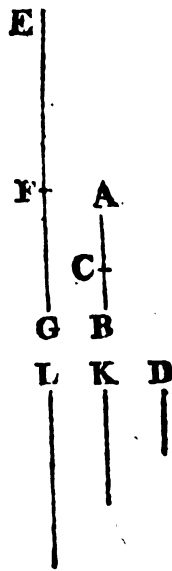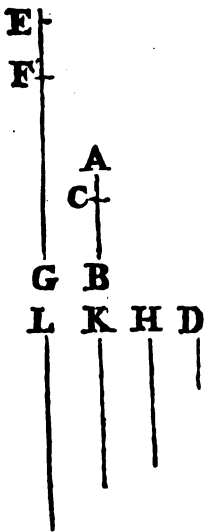
and, by the conftruc-
tion, EF is greater than
D ; therefore the whole
EG is greater than K and
D together : But K, to-
gether with D, is equal
to L ; therefore EG is
greater than L ; but FG
is not greater than L ;
and EG, FG are equi-
multiples of AB, BC,
and L is a multiple of
D ; therefore b AB has
to D a greater ratio than
BC has to D.

Alfo D has to BC a
greater ratio than it has
to AB : For, having
made the fame con-
ftruction, it may be
fhown, in like manner,
that L is greater than

b 7. def. 5.

FG, but that it is not greater than EG : And L is a multiple of
D ; and FG, EG are equimultiples of CB, AB; therefore D
has to CB a greater ratio b than it has to AB. Wherefore, of
unequal magnitudes, &c. Q. E. D.

PROP. IX. THEOR.

MAGNITUDES which have the fame ratio to the fame magnitude are equal to one another; and thofe to which the fame magnitude has the fame ratio are equal to one another.

Let A, B have each of them the fame ratio to C; A is e-qual to B: For, if they are not equal, one of them is greater than the other; let A be the greater; then, by what was fhown in the preceding propofition, there are fome equimultiples of A and B, and fome multiple of C fuch, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than that of C. Let fuch multiples be taken, and let D, E, be the equimultiples of A, B, and F the multiple of C, fo that D may be greater than F, and E not greater than F: But, becaufe A is to C, as B is to C, and of A, B, are taken equimultiples D, E, and of C is taken a multiple F; and that D is greater than F; E fhall alfo be great-er than F[a]; but E is not greater than F, which is impoffible; A therefore and B are not unequal; that is, they are equal.

a 5. def. 5.

Next, Let C have the fame ratio to each of the magnitudes A and B; A is equal to B: For, if they are not, one of them is greater than the other; let A be the greater; therefore, as was fhown in Prop. 8th, there is fome multiple F of C, and fome equimultiples E and D of B and A fuch, that F is greater than E, and not greater than D; but becaufe C is to B, as C is to A, and that F, the multiple of the firft, is greater than E, the multiple of the fecond; F the mul-tiple of the third, is greater than D, the multiple of the fourth[a]; But F is not greater than D, which is impoffible. Therefore A is equal to B. Wherefore magnitudes which, &c. Q. E. D.
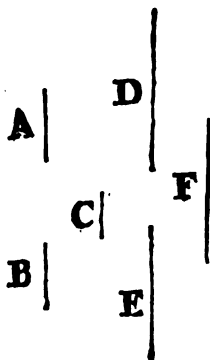
PROP. X. THEOR.

THAT magnitude which has a greater ratio than an- See N.
other has unto the fame magnitude is the greater
of the two : And that magnitude to which the fame has
a greater ratio than it has unto another magnitude is the
leffer of the two.

Let A have to C a greater ratio than B has to C : A is great-
er than B : For, becaufe A has a greater ratio to C, than B
has to C, there are ᵃ fome equimultiples of A and B, and ᵃ 7. Def. 5.
fome multiple of C fuch, that the multiple of A is greater than
the multiple of C, but the multiple of B is not greater than it :
Let them be taken, and let D, E be equi-
multiples of A, B, and F a multiple of C
fuch, that D is greater than F, but E is
not greater than F : Therefore D is greater
than E : And, becaufe D and E are equi-
multiples of A and B, and D is greater
than E ; therefore A is ᵇ greater than B.
Next, Let C have a greater ratio to B
than it has to A ; B is lefs than A : For ᵃ
there is fome multiple F of C, and fome
equimultiples E and D of B and A fuch,
that F is greater than E, but is not greater
than D : E therefore is lefs than D ; and
becaufe E and D are equimultiples of B
and A, therefore B is ᵇ lefs than A. That
magnitude, therefore, &c. Q. E. D.

D

A

C   F   b 4. Ax. 5.

B

E
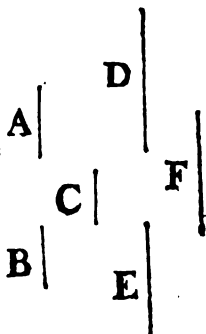
PROP. XI. THEOR.

RATIOS that are the fame to the fame ratio, are the
fame to one another.

Let A be to B, as C is to D ; and as C to D, fo let E be to
F ; A is to B, as E to F.
Take of A, C, E, any equimultiples whatever G, H, K ; and
of B, D, F, any equimultiples whatever L, M, N. Therefore,
fince A is to B, as C to D, and G, H are taken equimultiples of
I 4   A,

A, C, and L, M of B, D; if G be greater than L, H is greater than M; and if equal, equal; and if lefs, lefs [a]. Again, be-caufe C is to D, as E is to F, and H, K are taken equimultiples of C, E; and M, N, of D, F; if H be greater than M, K is greater than N; and if equal, equal; and if lefs, lefs: But, if G

G ——————— H ——————— K ——————————

A ——— C ——— E ———

B ——— D ——— F ———

L ——————— M ——————— N ——————————

be greater than L, it has been fhown that H is greater than M; and if equal, equal; and if lefs, lefs; therefore, if G be greater than L, K is greater than N; and if equal, equal; and if lefs, lefs: And G, K, are any equimultiples whatever of A, E; and L, N any whatever of B, F: Therefore, as A is to B, fo is E to F [a]. Wherefore ratios that, &c. Q. E. D.

P R O P. XII.　T H E O R.

IF any number of magnitudes be proportionals, as one of the antecedents is to its confequent, fo fhall all the antecedents taken together be to all the confequents.

Let any number of magnitudes A, B, C, D, E, F, be propor-tionals; that is, as A is to B, fo C to D, and E to F: As A is to B, fo fhall A, C, E together be to B, D, F together.

Take of A, C, E any equimultiples whatever G, H, K;

G ——————— H ——————— K ——————————

A ——— C ——— E ———

B ——— D ——— F ———

L ——————— M ——————— N ——————————

and of B, D, F any equimultiples whatever L, M, N: Then, becaufe A is to B, as C is to D, and as E to F; and that G, H,

　　　　　　　　　　　　　　　　　　　　　　　　K

K are equimultiples of A, C, E, and L, M, N equimultiples of B, D, F; if G be greater than L, H is greater than M, and K greater than N; and if equal, equal; and if lefs, lefs[a]. Wherefore, if G be greater than L, then G, H, K together are greater than L, M, N together; and if equal, equal; and if lefs, lefs. And G, and G, H, K together are any equimultiples of A, and A, C, E together; becaufe, if there be any number of magnitudes equimultiples of as many, each of each, whatever multiple one of them is of its part, the fame multiple is the whole of the whole[b]: For the fame reafon L, and L, M, N are any equimultiples of B, and B, D, F: As therefore A is to B, fo are A, C, E together to B, D, F together. Wherefore, if any number, &c. Q. E. D.

PROP. XIII. THEOR.

IF the firft has to the fecond the fame ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the fixth; the firft fhall alfo have to the fecond a greater ratio than the fifth has to the fixth.

Let A the firft, have the fame ratio to B the fecond, which C the third, has to D the fourth, but C the third, to D the fourth, a greater ratio than E the fifth, to F the fixth: Alfo the firft A fhall have to the fecond B a greater ratio than the fifth E to the fixth F.

Becaufe C has a greater ratio to D, than E to F, there are fome equimultiples of C and E, and fome of D and F fuch, that the multiple of C is greater than the multiple of D, but

M————	G—————	H————————
A———	C———	E———
B———	D———	F———
N————	K————	L—————————
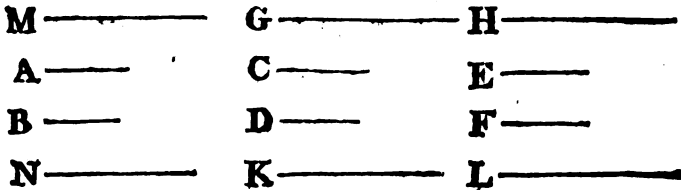
the multiple of E is not greater than the multiple of F[a]: Let fuch be taken, and of C, E let G, H be equimultiples, and K, L equimultiples of D, F, fo that G be greater than K, but H not greater than L; and whatever multiple G is of C, take M the fame multiple of A; and what multiple K is of D, take N the fame multiple of B: then, becaufe A is to B, as C to D, and

of

of A and C, M and G are equimultiples: And of B and D, N and K are equimultiples; if M be greater than N, G is greater

b 5. def. 5. than K; and if equal, equal; and if lefs, lefs [b]; but G is greater than K, therefore M is greater than N: But H is not greater than L; and M, H are equimultiples of A, E; and N, L equimultiples of B, F: Therefore A has a greater ratio

a 7. def. 5. to B, than E has to F [c]. Wherefore, if the firft, &c. Q. E. D.

Cor. And if the firft has a greater ratio to the fecond, than the third has to the fourth, but the third the fame ratio to the fourth, which the fifth has to the fixth; it may be demonftrated in like manner, that the firft has a greater ratio to the fecond, than the fifth has to the fixth.

PROP. XIV. THEOR.

See N.   IF the firft has to the fecond, the fame ratio which the third has to the fourth; then, if the firft be greater than the third, the fecond fhall be greater than the fourth; and if equal, equal; and if lefs, lefs.

Let the firft A, have to the fecond B, the fame ratio which the third C, has to the fourth D; if A be greater than C, B is greater than D.

a 8. 5. Becaufe A is greater than C, and B is any other magnitude, A has to B a greater ratio than C to B [a]: But, as A is to B, fo
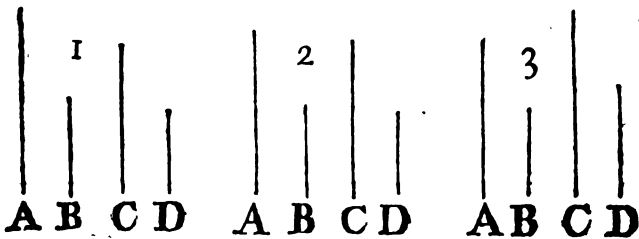
is C to D; therefore alfo C has to D a greater ratio than C has

b 13. 5. to B [b]: But of two magnitudes, that to which the fame has the greater ratio is the leffer [c]: Wherefore D is lefs than B; that

c 10. 5. is, B is greater than D.

Secondly, If A be equal to C, B is equal to D: For A is to

d 9. 5. B, as C, that is A, to D; B therefore is equal to D [d].

Thirdly, If A be lefs than C, B fhall be lefs than D: For C is greater than A, and becaufe C is to D, as A is to B, D is greater than B, by the firft cafe; wherefore B is lefs than D. Therefore, if the firft, &c. Q. E. D.

<div align="right">PROP.</div>

PROP. XV. THEOR.

MAGNITUDES have the fame ratio to one another which their equimultiples have.

Let AB be the fame multiple of C, that DE is of F: C is to F, as AB to DE.

Becaufe AB is the fame multiple of C, that DE is of F; there are as many magnitudes in AB equal to C, as there are in DE equal to F: Let AB be divided into magnitudes, each equal to C, viz. AG, GH, HB; and DE into magnitudes, each equal to F, viz. DK, KL, LE: Then the number of the firft AG, GH, HB, fhall be equal to the number of the laft DK, KL, LE: And becaufe AG, GH, HB are all equal, and that DK, KL, LE are alfo equal to one another; therefore AG is to DK, as GH to KL, and as HB to LE[a]: And as one of the antecedents to its confequent, fo are all the antecedents together to all the confequents together[b]; wherefore, as AG is to DK, fo is AB to DE: But AG is equal to C, and DK to F: Therefore, as C is to F, fo is AB to DE. Therefore magnitudes, &c. Q. E. D.

a 7. 5.

b 12. 5.
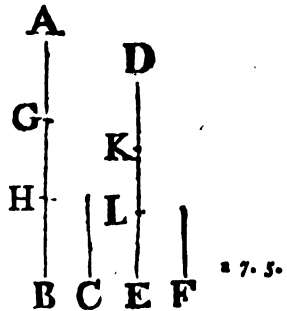
PROP. XVI.   THEOR.

IF four magnitudes of the fame kind be proportionals, they fhall alfo be proportionals when taken alternately.

Let the four magnitudes A, B, C, D be proportionals, viz. as A to B, fo C to D: They fhall alfo be proportionals when taken alternately; that is, A is to C, as B to D.

Take of A and B any equimultiples whatever E and F; and of C and D take any equimultiples whatever G and H: And

because

becaufe E is the fame multiple of A, that F is of B, and that magnitudes have the fame ratio to one another which their equimultiples have [a]; therefore A is to B, as E is to F: But as A is to B, fo is C to D: Wherefore, as C is to D, fo [b] is E to F: Again, becaufe G, H are equimultiples of C, D, as C is to D, fo is G to H [a]; but as C is to D, fo is E to F. Wherefore, as E is to F, fo is G to H [b]. But, when four magnitudes are proportionals, if the firft be greater than the third, the fecond fhall be greater than the fourth; and if equal, equal; if lefs, lefs [c]. Wherefore, if E be greater than G, F likewife is greater than H; and if equal, equal; if lefs, lefs: And E, F are any equimultiples whatever of A, B; and, G, H any whatever of C, D. Therefore A is to C, as B to D [d]. If then four magnitudes, &c. Q. E. D.
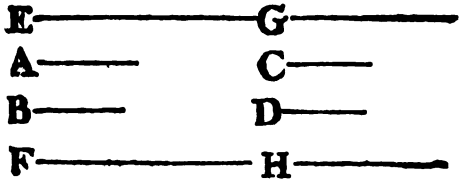
E ———————————— G ————————————

A ——————— C ————

B ——————— D ————

F ———————————— H ————————————

PROP. XVII. THEOR.

IF magnitudes, taken jointly, be proportionals, they fhall alfo be proportionals when taken feparately; that is, if two magnitudes together have to one of them the fame ratio which two others have to one of thefe, the remaining one of the firft two fhall have to the other the fame ratio which the remaining one of the laft two has to the other of thefe.

Let AB, BE, CD, DF be the magnitudes taken jointly which are proportionals; that is, as AB to BE, fo is CD to DF; they fhall alfo be proportionals taken feparately, viz. as AE to EB, fo CF to FD.

Take of AE, EB, CF, FD any equimultiples whatever GH, HK, LM, MN; and again, of EB, FD, take any equimultiples whatever KX, NP: And becaufe GH is the fame multiple of AE, that HK is of EB, therefore GH is the fame multiple [a] of AE, that GK is of AB: But GH is the fame multiple of AE, that LM is of CF; wherefore GK is the fame multiple of AB,

that

that LM is of CF. Again, becaufe LM is the fame multiple of
CF, that MN is of FD; therefore LM is the fame multiple ᵃ of
CF, that LN is of CD: But LM was fhown to be the fame
multiple of CF, that GK is of AB; GK therefore is the fame
multiple of AB, that LN is of CD; that is, GK, LN are equi-
multiples of AB, CD. Next, becaufe HK is the fame multiple
of EB, that MN is of FD; and that KX is
alfo the fame multiple of EB, that NP is
of FD; therefore HX is the fame multiple
ᵇ of EB, that MP is of FD. And becaufe
AB is to BE, as CD is to DF, and that of
AB and CD, GK and LN are equimul-
tiples, and of EB and FD, HX and MP are
equimultiples; if GK be greater than HX,
then LN is greater than MP; and if equal,
equal; and if lefs, lefs ᶜ: But if GH be
greater than KX, by adding the common
part HK to both, GK is greater than HX;
wherefore alfo LN is greater than MP;
and by taking away MN from both, LM
is greater than NP: Therefore, if GH be
greater than KX, LM is greater than NP.
In like manner it may be demonftrated,
that if GH be equal to KX, LM likewife is equal to NP; and
if lefs, lefs: And GH, LM are any equimultiples whatever of
AE, CF, and KX, NP are any whatever of EB, FD. There-
fore ᶜ, as AE is to EB, fo is CF to FD. If then magnitudes,
&c. Q. E. D.

ᵃ 1. 5.

ᵇ 2. 5.

ᶜ 5. def. 5.
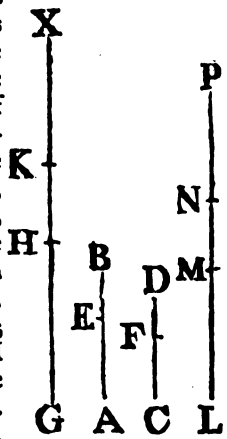
P R O P. XVIII.   T H E O R.

IF magnitudes, taken feparately, be proportionals, they
fhall alfo be proportionals when taken jointly, that is,
if the firft be to the fecond, as the third to the fourth,
the firft and fecond together fhall be to the fecond, as
the third and fourth together to the fourth.

Let AE, EB, CF, FD be proportionals; that is, as AE to
EB, fo is CF to FD; they fhall alfo be proportionals when
taken jointly; that is, as AB to BE, fo CD to DF.
Take of AB, BE, CD, DF any equimultiples whatever GH,
HK, LM, MN; and again, of BE, DF, take any whatever e-
quimultiples KO, NP: And becaufe KO, NP are equimultiples
of

of BE, DF; and that KH, NM are equimultiples likewife of BE, DF, if KO, the multiple of BE, be greater than KH, which is a multiple of the fame BE, NP, likewife the multiple of DF, fhall be greater than NM, the multiple of the fame DF; and if KO be equal to KH, NP fhall be equal to NM; and if lefs, lefs.

Firft, Let KO not be greater than KH, therefore NP is not greater than NM: And becaufe GH, HK are equi-multiples of AB, BE, and that AB is greater than BE, therefore GH is **a. 3. Ax.** greater [a] than HK; but KO is not greater than KH, wherefore GH is greater than KO. In like manner it may be fhown, that LM is greater than NP. Therefore, if KO be not great-er than KH, then GH, the multiple of AB, is always greater than KO, the multiple of BE; and likewife LM, the multiple of CD, greater than NP, the multiple of DF.
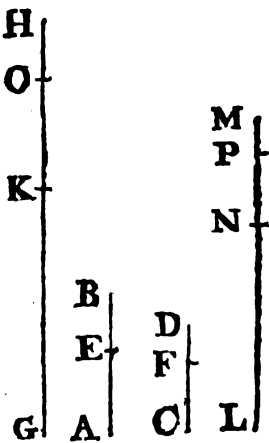
Next, Let KO be greater than KH; therefore, as has been fhown, NP is greater than NM: And becaufe the whole GH is the fame multiple of the whole AB, that HK is of BE, the re-mainder GK is the fame multiple of **b 5. 5.** the remainder AE, that GH is of AB [b]: which is the fame that LM is of CD. In like manner, becaufe LM is the fame multiple of CD, that MN is of DF, the remainder LN is the fame multiple of the remainder CF, that the whole LM is of the whole CD [b]: But it was fhown that LM is the fame multiple of CD, that GK is of AE; therefore GK is the fame multiple of AE, that LN is of CF; that is, GK, LN are equimultiples of AE, CF: And becaufe KO, NP are equimul-tiples of BE, DF, if from KO, NP there be taken KH, NM, which are likewife equimultiples of BE, DF, the remainders HO, MP are either equal to BE, **c 6. 5.** DF, or equimultiples of them [c]. Firft, Let HO, MP, be e-qual to BE, DF; and becaufe AE is to EB, as CF to FD, and that

that GK, LN are equimultiples of AE, CF; GK shall be to EB, as LN to FD [d] : But HO is equal to EB, and MP to FD; wherefore GK is to HO, as LN to MP. If therefore GK be greater than HO, LN is greater than MP; and if equal, equal; and if less [e], less.

But let HO, MP be equimultiples of EB, FD; and because AE is to EB, as CF to FD, and that of AE, CF are taken equimultiples GK, LN; and of EB, FD, the equimultiples HO, MP; if GK be greater than HO, LN is greater than MP; and if equal, equal; and if less, less [f]; which was likewise shown in the preceding case. If therefore GH be greater than KO, taking KH from both, GK is greater than HO; wherefore also LN is greater than MP; and consequently, adding NM to both, LM is greater than NP: Therefore, if GH be greater than KO, LM is greater than NP. In like manner it may be shown, that if GH be equal to KO, LM is equal to NP; and if less, less. And in the case in which KO is not greater than KH, it has been shown that GH is always greater than KO, and likewise LM than NP: But GH, LM are any equimultiples of AB, CD, and KO, NP are any whatever of BE, DF; therefore [f], as AB is to BE, so is CD to DF. If then magnitudes, &c. Q. E. D.
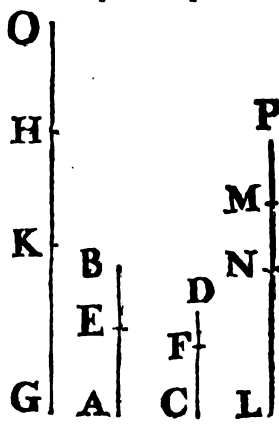
P R O P. XIX. T H E O R.

IF a whole magnitude be to a whole, as a magnitude taken from the first, is to a magnitude taken from the other; the remainder shall be to the remainder, as the whole to the whole.

Let the whole AB, be to the whole CD, as AE, a magnitude taken from AB, to CF, a magnitude taken from CD; the remainder EB, shall be to the remainder FD, as the whole AB to the whole CD.

Because AB is to CD, as AE to CF; likewise, alternately [a], a 16. 5.

BA

BA is to AE, as DC to CF: And becaufe, if magnitudes, taken jointly, be proportionals, they are alfo proportionals [b] when taken feparately ; therefore, as BE is to EA, fo is DF to FC ; and alternately, as BE is to DF, fo is EA to FC: But, as AE to CF, fo, by the hypothefis, is AB to CD ; therefore alfo BE, the remainder, fhall be to the remainder DF, as the whole AB to the whole CD: Wherefore, if the whole, &c. Q. E. D.

Cor. If the whole be to the whole, as a magnitude taken from the firft, is to a magnitude taken from the other ; the remainder likewife is to the remainder, as the magnitude taken from the firft to that taken from the other : The demonftration is contained in the preceding.
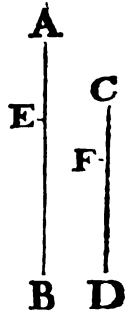
PROP. E. THEOR.

IF four magnitudes be proportionals, they are alfo proportionals by converfion, that is, the firft is to its excefs above the fecond, as the third to its excefs above the fourth.

Let AB be to BE, as CD to DF ; then BA is to AE, as DC to CF.

Becaufe AB is to BE, as CD to DF, by divifion [a], AE is to EB, as CF to FD ; and by inverfion [b], BE is to EA, as DF to FC. Wherefore, by compofition [c], BA is to AE, as DC is to CF: If, therefore, four, &c. Q. E. D.

PROP. XX. THEOR.

IF there be three magnitudes, and other three, which, taken two and two, have the fame ratio ; if the firft be greater than the third, the fourth fhall be greater than the fixth ; and if equal, equal ; and if lefs, lefs.

Let

Let A, B, C be three magnitudes, and D, E, F other three, which, taken two and two, have the same ratio, viz. as A is to B, so is D to E; and as B to C, so is E to F. If A be greater than C, D shall be greater than F; and, if equal, equal; and if less, less.

Because A is greater than C; and B is any other magnitude, and that the greater has to the same magnitude a greater ratio than the less has to it [a]; therefore A has to B a greater ratio than C has to B: But as D is to E, so is A to B; therefore [b] D has to E a greater ratio than C to B: And because B is to C, as E to F, by inversion, C is to B, as F is to E; and D was shown to have to E a greater ratio than C to B; therefore D has to E a greater ratio than F to E [c]: But the magnitude which has a greater ratio than another to the same magnitude, is the greater of the two [d]: D is therefore greater than F.

a 8. 5.

b 13. 5.

c Cor. 13. 5.

d 10. 5.
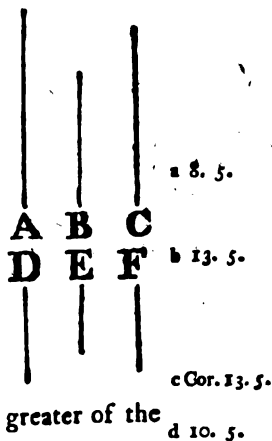
Secondly, Let A be equal to C; D shall be equal to F: Because A and C are equal to one another, A is to B, as C is to B [e]: But A is to B, as D to E; and C is to B, as F to E; wherefore D is to E, as F to E [f]; and therefore D is equal to F [g].
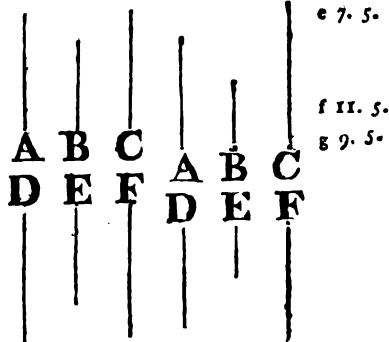
Next, Let A be less than C; D shall be less than F: For C is greater than A, and, as was shown in the first case, C is to B, as F to E, and in like manner B is to A, as E to D; therefore F is greater than D, by the first case; and therefore D is less than F. Therefore, if there be three, &c. Q. E. D.

e 7. 5.

f 11. 5.

g 9. 5.

PROP. XXI. THEOR.

IF there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

K

Let

Let A, B, C be three magnitudes, and D, E, F other three, which have the fame ratio, taken two and two, but in a crofs order, viz. as A is to B, fo is E to F, and as B is to C, fo is D to E. If A be greater than C, D fhall be greater than F; and if equal, equal; and if lefs, lefs.

Becaufe A is greater than C, and B is any other magnitude, A has to B a greater ratio[a] than C has to B: But as E to F, fo is A to B; therefore[b] E has to F a greater ratio than C to B: And becaufe B is to C, as D to E, by inverfion, C is to B, as E to D: And E was fhown to have to F a greater ratio than C to B; therefore E has to F a greater ratio than E to D[c]; but the magnitude to which the fame has a greater ratio than it has to another, is the leffer of the two[d]; F therefore is lefs than D; that is, D is greater than F.

a 8. 5.

b 13. 5.

c Cor. 13. 5.

d 10. 5.

$$A \quad B \quad C$$
$$D \quad E \quad F$$

Secondly, Let A be equal to C; D fhall be equal to F. Becaufe A and C are equal, A is[e] to B, as C is to B: But A is to B, as E to F; and C is to B, as E to D; wherefore E is to F as E to D[f]; and therefore D is equal to F[g].

e 7. 5.

f 11. 5.

g 9. 5.

Next, Let A be lefs than C; D fhall be lefs than F: For C is greater than A, and, as was fhown, C is to B, as E to D, and in like manner B is to A, as F to E; therefore F is greater than D, by cafe firft; and therefore D is lefs than F. Therefore, if there be three, &c. Q. E. D.

$$A \quad B \quad C \qquad A \quad B \quad C$$
$$D \quad E \quad F \qquad D \quad E \quad F$$

PROP. XXII. THEOR.

IF there be any number of magnitudes, and as many others, which, taken two and two in order, have the fame ratio; the firft fhall have to the laft of the firft magnitudes the fame ratio which the firft of the others has to the laft. N. B. *This is ufually cited by the words* " *ex aequali,*" *or* " *ex aequo.*"

Firft,

First, Let there be three magnitudes A, B, C, and as many others D, E, F, which, taken two and two, have the same ratio, that is, such that A is to B as D to E; and as B is to C, so is E to F; A shall be to C, as D to F.

Take of A and D any equimultiples whatever G and H; and of B and E any equimultiples whatever K and L; and of C and F any whatever M and N: Then, because A is to B, as D to E, and that G, H are equimultiples of A, D, and K, L equimultiples of B, E; as G is to K, so is ª H to L: For the same reason, K is to M, as L to N: And because there are three magnitudes G, K, M, and other three H, L, N, which, two and two, have the same ratio; if G be greater than M, H is greater than N; and if equal, equal; and if less, less ᵇ: And G, H are any equimultiples whatever of A, D, and M, N are any equimultiples whatever of C, F: Therefore ᶜ, as A is to C, so is D to F.

a 4. 5.

b 20. 5.

c 5. def. 5.

A B C D E F
G K M H L N
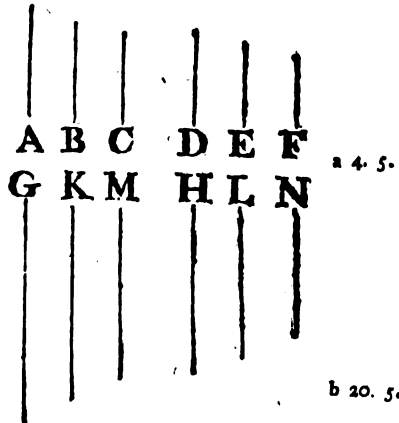
Next, Let there be four magnitudes A, B, C, D, and other four E, F, G, H, which two and two have the same ratio, viz. as A is to B, so is E to F; and as B to C, so F to G; and as C to D, so G to H: A shall be to D, as E to H.

| A. B. C. D. |
| E. F. G. H. |
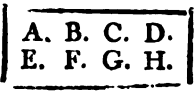
Because A, B, C are three magnitudes, and E, F, G other three, which, taken two and two, have the same ratio; by the foregoing case, A is to C, as E to G: But C is to D, as G is to H; wherefore again, by the first case, A is to D, as E to H; and so on, whatever be the number of magnitudes. Therefore, if there be any number, &c. Q. E. D.

PROP.

PROP. XXIII.   THEOR.

IF there be any number of magnitudes, and as many others, which, taken two and two, in a crofs order, have the fame ratio ; the firft fhall have to the laft of the firft magnitudes the fame ratio which the firft of the others has to the laft.   N. B. *This is ufually cited by the words " ex aequali in proportione perturbata ;" or " ex aequo perturbate."*

Firft, Let there be three magnitudes A, B, C, and other three D, E, F, which, taken two and two, in a crofs order, have the fame ratio, that is, fuch that A is to B, as E to F; and as B is to C, fo is D to E : A is to C, as D to F.

Take of A, B, D any equimultiples whatever G, H, K ; and of C, E, F any equimultiples whatever L, M, N: And be-caufe G, H are equimultiples of A, B, and that magnitudes have the fame ratio which their equi-

a 15. 5.   multiples have ᵃ; as A is to B, fo is G to H: And for the fame reafon, as E is to F, fo is M to N: But as A is to B, fo is E to F; as therefore G is to H, fo is M

b 11. 5.   to Nᵇ. And becaufe as B is to C, fo is D to E, and that H, K are equimultiples of B, D, and L, M of C, E ; as H is to L, fo is

c 4. 5.   ᶜ K to M: And it has been fhown that G is to H, as M to N : Then, becaufe there are three magni-tudes G, H, L, and other three K, M, N which have the fame ratio taken two and two in a crofs order ; if G be greater than L,

d 21. 1.   K is greater than N ; and if equal, equal ; and if lefs, lefs ᵈ; and G, K are any equimultiples whatever of A, D ; and L, N any whatever of C, F ; as, therefore, A is to C, fo is D to F.
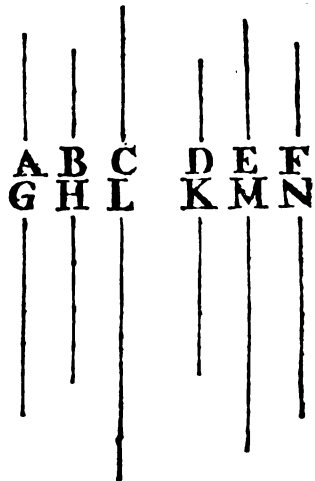
A B C   D E F
G H L   K M N

Next,

Next, Let there be four magnitudes, A, B, C, D, and o-
ther four E, F, G, H, which, taken two and
two in a crofs order, have the fame ratio, viz.
A to B, as G to H; B to C, as F to G; and
C to D, as E to F: A is to D, as E to H.

A. B	C D.
E. F.	G. H.

Becaufe A, B, C are three magnitudes, and F, G, H other
three, which, taken two and two in a crofs order, have the
fame ratio; by the firft cafe, A is to C, as F to H: But C is
to D, as E is to F; wherefore again, by the firft cafe, A is to
D, as E to H: And fo on, whatever be the number of magni-
tudes. Therefore, if there be any number, &c. Q. E. D.

P R O P. XXIV. T H E O R.

IF the firft has to the fecond the fame ratio which the
third has to the fourth; and the fifth to the fecond,
the fame ratio which the fixth has to the fourth; the
firft and fifth together fhall have to the fecond, the fame
ratio which the third and fixth together have to the
fourth.

Let AB the firft, have to C the fecond, the fame ratio which
DE the third, has to F the fourth; and let BG the fifth, have
to C the fecond, the fame ratio which EH
the fixth, has to F the fourth: AG, the
firft and fifth together, fhall have to C the
fecond, the fame ratio which DH, the
third and fixth together, has to F the
fourth.

Becaufe BG is to C, as EH to F; by in-
verfion, C is to BG, as F to EH: And be-
caufe, as AB is to C, fo is DE to F; and
as C to BG, fo F to EH; ex aequali ᵃ, AB
is to BG, as DE to EH: And becaufe
thefe magnitudes are proportionals, they
fhall likewife be proportionals when taken
jointly ᵇ; as therefore AG is to GB, fo is
DH to HE; but as GB to C, fo is HE to F. Therefore, ex
aequali ᵃ, as AG is to C, fo is DH to F. Wherefore, if the
firft, &c. Q. E. D.
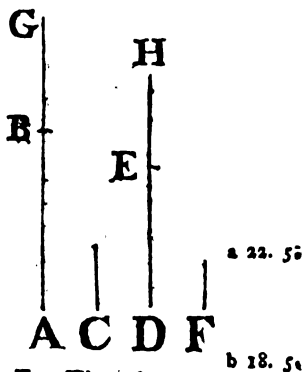
COR. 1. If the fame hypothefis be made as in the propofi-
tion, the excefs of the firft and fifth fhall be to the fecond, as

a 22. 5.

b 18. 5.

K 3   the

the excefs of the third and fixth to the fourth : The demonftration of this is the fame with that of the propofition, if divifion be ufed inftead of compofition.

Cor. 2. The propofition holds true of two ranks of magnitudes, whatever be their number, of which each of the firft rank has to the fecond magnitude the fame ratio that the correfponding one of the fecond rank has to a fourth magnitude ; as is manifeft.

PROP. XXV. THEOR.

IF four magnitudes of the fame kind are proportionals, the greateft and laeft of them together are greater than the other two together.

Let the four magnitudes AB, CD, E, F be proportionals, viz. AB to CD, as E to F; and let AB be the greateft of them,

a A. & 14. and confequently F the leaft ᵃ. AB, together with F, are greater
15. than CD, together with E.

Take AG equal to E, and CH equal to F: Then, becaufe as AB is to CD, fo is E to F, and that AG is equal to E, and CH equal to F; AB is to CD, as AG to CH. And becaufe AB the whole, is to the whole CD, as AG is to CH, likewife the remainder GB fhall be to the remainder
b 19. 5. HD, as the whole AB is to the whole ᵇ CD : But AB is greater than CD, therea A. 5. foreᶜ GB is greater than HD : And becaufe AG is equal to E, and CH to F ; AG and F together are equal to CH and E together. If therefore to the unequal magnitudes GB, HD, of which GB is the greater, there be added equal magnitudes, viz. to GB the two AG and F, and CH and E to HD ; AB and F together are greater than CD and E. Therefore, if four magnitudes, &c. Q. E. D.
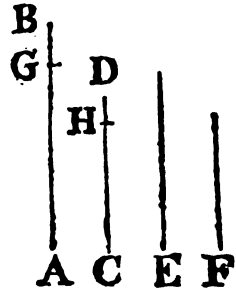
PROP. F. THEOR.

RATIOS which are compounded of the fame ratios, are the fame with one another.

Let

Let A be to B, as D to E; and B to C, as E to F: The ra-
tio which is compounded of the ratios of
A to B, and B to C, which, by the definition
of compound ratio, is the ratio of A to C, is
the same with the ratio of D to F, which, by
the same definition, is compounded of the
ratios of D to E, and E to F.

> A. B. C.
> D. E. F.

Because there are three magnitudes A, B, C, and three o-
thers D, E, F, which, taken two and two in order, have the
same ratio; ex aequali, A is to C, as D to F a.

Next, Let A be to B, as E to F, and B to C, as D to to E;
therefore, *ex aequali in proportione perturbata*
b, A is to C, as D to F; that is, the ratio of A
to C, which is compounded of the ratios of A to
B, and B to C, is the same with the ratio of D
to F, which is compounded of the ratios of D
to E, and E to F: And in like manner the proposition may be
demonstrated, whatever be the number of ratios in either case.

> A. B. C.
> D. E. F.

PROP. G. THEOR.

IF several ratios be the same with several ratios, each to
each; the ratio which is compounded of ratios which
are the same with the first ratios, each to each, is the
same with the ratio compounded of ratios which are the
same with the other ratios, each to each.

Let A be to B, as E to F; and C to D, as G to H: And let
A be to B, as K to L; and C to D, as L to M: Then the ra-
tio of K to M, by the definition
of compound ratio, is compound-
ed of the ratios of K to L, and
L to M, which are the same with
the ratios of A to B, and C to D:

> A. B. C. D.   K. L. M.
> E. F. G. H.   N. O. P.

And as E to F, so let N be to O; and as G to H, so let O be to
P; then the ratio of N to P is compounded of the ratios of N to
O, and O to P, which are the same with the ratios of E to F,
and G to H: And it is to be shown that the ratio of K to M, is
the same with the ratio of N to P, or that K is to M, as N to P.

Because K is to L, as (A to B, that is, as E to F, that is, as)
N to O; and as L to M, so is (C to D, and so is G to H,

K 4   and

and fo is) O to P: Ex aequali[a], K is to M, as N to P. There,
fore, if feveral ratios, &c. Q. E. D.

a 22. 5.

PROP. H. THEOR.

See N.

IF a ratio compounded of feveral ratios be the fame
with a ratio compounded of any other ratios, and if
one of the firft ratios, or a ratio compounded of any of
the firft, be the fame with one of the laft ratios, or with
the ratio compounded of any of the laft ; then the ratio
compounded of the remaining ratios of the firft, or the
remaining ratio of the firft, if but one remain, is the
fame with the ratio compounded of thofe remaining of
the laft, or with the remaining ratio of the laft.

Let the firft ratios be thofe of A to B, B to C, C to D,
D to E, and E to F; and let the other ratios be thofe of G to
H, H to K, K to L, and L to M : Alfo, let the ratio of A to

a Definition F, which is compounded of[a] the firft
of com- ratios be the fame with the ratio of G
pounded to M, which is compounded of the
ratio. other ratios : And befides, let the ra-

| A. B. C. D. E. F. |
| G. H. K. L. M. |

tio of A to D, which is compounded of the ratios of A to B,
B to C, C to D, be the fame with the ratio of G to K, which is
compounded of the ratios of G to H, and H to K : Then the
ratio compounded of the remaining firft ratios, to wit, of the
ratios of D to E, and E to F, which compounded ratio is the
ratio of D to F, is the fame with the ratio of K to M, which is
compounded of the remaining ratios of K to L, and L to M of
the other ratios.

Becaufe, by the hypothefis, A is to D, as G to K, by in-
b B. 5. verfion[b], D is to A, as K to G ; and as A is to F, fo is G to
c 22. 5. M ; therefore[c], ex aequali, D is to F, as K to M. If therefore
a ratio which is, &c. Q. E. D.

PROP. K. THEOR.

IF there be any number of ratios, and any number of other ratios such, that the ratio compounded of ratios which are the same with the first ratios, each to each, is the same with the ratio compounded of ratios which are the same, each to each, with the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios which are the same with several of the first ratios, each to each, be the same with one of the last ratios, or with the ratio compounded of ratios which are the same, each to each, with several of the last ratios: Then the ratio compounded of ratios which are the same with the remaining ratios of the first, each to each, or the remaining ratio of the first, if but one remain; is the same with the ratio compounded of ratios which are the same with those remaining of the last, each to each, or with the remaining ratio of the last.

Let the ratios of A to B, C to D, E to F be the first ratios; and the ratios of G to H, K to L, M to N, O to P, Q to R, be the other ratios: And let A be to B, as S to T; and C to D, as T to V; and E to F, as V to X: Therefore, by the definition of compound ratio, the ratio of S to X is compounded

$$
\begin{array}{lll}
& \text{h, k, l.} & \\
\text{A, B; C, D; E, F.} & & \text{S, T, V, X.} \\
\text{G, H; K, L; M, N; O, P; Q, R.} & & \text{Y, Z, a, b, c, d.} \\
\text{e, f, g.} & \text{m, n, o, p.} &
\end{array}
$$

of the ratios of S to T, T to V, and V to X, which are the same with the ratios of A to B, C to D, E to F, each to each: Also, as G to H, so let Y be to Z; and K to L, as Z to a; M to N, as a to b, O to P, as b to c; and Q to R, as c to d: Therefore, by the same definition, the ratio of Y to d is compounded of the ratios of Y to Z, Z to a, a to b, b to c, and

c to

e to d, which are the fame, each to each, with the ratios of G to H, K to L, M to N, O to P, and Q to R : Therefore, by the hypothefis, S is to X, as Y to d : Also, let the ratio of A to B, that is, the ratio of S to T, which is one of the firft ratios, be the fame with the ratio of e to g, which is compounded of the ratios of e to f, and f to g, which, by the hypothefis, are the fame with the ratios of G to H, and K to L, two of the other ratios ; and let the ratio of h to l be that which is compounded of the ratios of h to k, and k to l, which are the fame with the remaining firft ratios, viz. of C to D, and E to F; also, let the ratio of m to p be that which is compounded of the ratios of m to n, n to o, and o to p, which are the fame, each to each, with the remaining other ratios, viz. of M to N, O to P, and Q to R : Then the ratio of h to l is the fame with the ratio of m to p, or h is to l, as m to p.

$$h, \quad k, \quad l.$$
$$A, B; C, D; E, F. \qquad\qquad S, T, V, X.$$
$$G, H; K, L; M, N; O, P; Q, R. \quad Y, Z, a, b, c, d.$$
$$e, f, g. \qquad\qquad m, n, o, p.$$

Becaufe e is to f, as (G to H, that is, as) Y to Z; and f is to g, as (K to L, that is, as) Z to a ; therefore, ex aequali, e is to g, as Y to a : And, by the hypothefis, A is to B, that is, S to T, as e to g; wherefore S is to T, as Y to a ; and, by inverfion, T is to S, as a to Y; and S is to X, as Y to d; therefore, ex aequali, T is to X, as a to d : Also, becaufe h is to k, as (C to D, that is, as) T to V ; and k is to l, as (E to F, that is, as) V to X ; therefore, ex aequali, h is to l, as T to X: In like manner, it may be demonftrated, that m is to p, as a to d : And it has been fhown, that T is to X, as a to d : There-
a 11. 5. fore a h is to l, as m to p.   Q. E. D.

The propofitions G and K are ufually, for the fake of brevity, expreffed in the fame terms with propofitions F and H : And therefore it was proper to fhow the true meaning of them when they are fo expreffed ; efpecially fince they are very frequently made ufe of by geometers.

THE

THE

ELEMENTS

OF

EUCLID.

BOOK VI.

DEFINITIONS,

I.

SIMILAR rectilineal figures are thofe which have their feveral angles equal, each to each, and the fides about the equal angles proportionals.

II.

" Reciprocal figures, viz. triangles and parallelograms, are See N.
" fuch as have their fides about two of their angles propor-
" tionals in fuch manner, that a fide of the firft figure is to
" a fide of the other, as the remaining fide of this other is to
" the remaining fide of the firft."

III.

A ftraight line is faid to be cut in extreme and mean ratio, when the whole is to the greater fegment, as the greater fegment is to the lefs.

IV.

The altitude of any figure is the ftraight line drawn from its vertex perpendicular to the bafe.
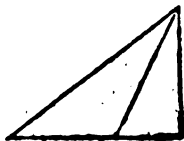
PROP.

PROP. I. THEOR.

TRIANGLES and parallelograms of the fame altitude are one to another as their bafes.

Let the triangles ABC, ACD, and the parallelograms EC, CF have the fame altitude, viz. the perpendicular drawn from the point A to BD: Then, as the bafe BC is to the bafe CD, fo is the triangle ABC to the triangle ACD, and the parallelogram EC to the parallelogram CF.

Produce BD both ways to the points H, L, and take any number of ftraight lines BG, GH, each equal to the bafe BC; and DK, KL, any number of them, each equal to the bafe CD; and join AG, AH, AK, AL: Then, becaufe CB, BG, GH are all equal, the triangles AHG, AGB, ABC are all equal[a]: Therefore, whatever multiple the bafe HC is of the bafe BC, the fame multiple is the triangle AHC of the triangle ABC: For the fame reafon, whatever multiple the bafe LC is of the bafe CD, the fame multiple is the triangle ALC of the triangle ADC: And if the bafe HC be equal to the bafe CL, the triangle AHC is alfo equal to the triangle ALC[a]; and if the bafe HC be greater than the

a 38. I.

bafe CL, likewife the triangle AHC is greater than the triangle ALC; and if lefs, lefs: Therefore, fince there are four magnitudes, viz. the two bafes BC, CD, and the two triangles ABC, ACD; and of the bafe BC and the triangle ABC, the firft and third, any equimultiples whatever have been taken, viz. the bafe HC and triangle AHC; and of the bafe CD and triangle ACD, the fecond and fourth, have been taken any equimultiples whatever, viz. the bafe CL and triangle ALC; and that it has been fhown, that, if the bafe HC be greater than the bafe CL, the triangle AHC is greater than the triangle ALC; and if equal, equal; and if lefs, lefs: Therefore[b], as the bafe BC is to the bafe CD, fo is the triangle ABC to the triangle ACD.
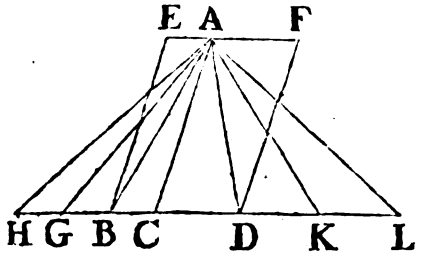
b 5. def. 5.

And becaufe the parallelogram CE is double of the triangle ABC,

ABC [a], and the parallelogram CF double of the triangle ACD, and that magnitudes have the fame ratio which their equimultiples have [d] ; as the triangle ABC is to the triangle ACD, fo is the parallelogram EC to the parallelogram CF: And becaufe it has been fhown, that, as the bafe BC is to the bafe CD, fo is the triangle ABC to the triangle ACD ; and as the triangle ABC to the triangle ACD, fo is the parallelogram EC to the parallelogram CF; therefore, as the bafe BC is to the bafe CD, fo is [e] the parallelogram EC to the parallelogram CF. Wherefore triangles, &c. Q. E. D.

c 41. 1.
d 15. 5.

e 11. 5.

Cor. From this it is plain, that triangles and parallelograms that have equal altitudes, are one to another as their bafes.

Let the figures be placed fo as to have their bafes in the fame ftraight line ; and having drawn perpendiculars from the vertices of the triangles to the bafes, the ftraight line which joins the vertices is parallel to that in which their bafes are [f], becaufe the perpendiculars are both equal and parallel to one another: Then, if the fame conftruction be made as in the propofition, the demonftration will be the fame.

f 33. 1.

P R O P. II.   T H E O R.

IF a ftraight line be drawn parallel to one of the fides of a triangle, it fhall cut the other fides, or thofe produced, proportionally : And if the fides, or the fides produced, be cut proportionally, the ftraight line which joins the points of fection fhall be parallel to the remaining fide of the triangle.

See N.

Let DE be drawn parallel to BC, one of the fides of the triangle ABC : BD is to DA, as CE to EA.

Join BE, CD ; then the triangle BDE is equal to the triangle CDE [a], becaufe they are on the fame bafe DE, and between the fame parallels DE, BC : ADE is another triangle, and equal magnitudes have to the fame, the fame ratio [b] ; therefore, as the triangle BDE to the triangle ADE, fo is the triangle CDE to the triangle ADE ; but as the triangle BDE to the triangle ADE, fo is [c] BD to DA, becaufe having the fame altitude, viz. the perpendicular drawn from the point E to AB, they are to one another as their bafes ; and for the fame

a 37. 1.

b 7. 5.

c 1. 6.

reafon,

reafon, as the triangle CDE to the triangle ADE, fo is CE to EA. Therefore, as BD to DA, fo is CE to EA ᵈ.

Next, Let the fides AB, AC of the triangle ABC, or thefe
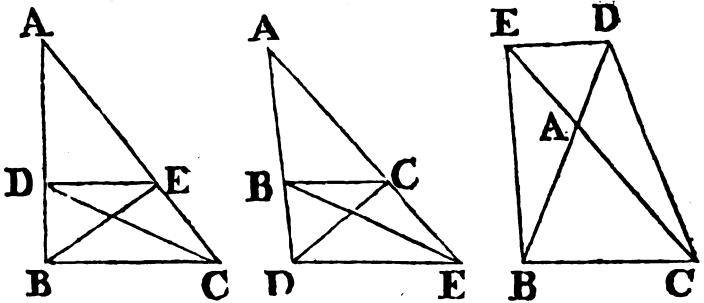
produced, be cut proportionally in the points D, E, that is, fo that BD be to DA, as CE to EA, and join DE : DE is parallel to BC.

The fame conftruction being made, becaufe as BD to DA, fo is CE to EA ; and as BD to DA, fo is the triangle BDE to
e 1. 6. the triangle ADE ᵉ; and as CE to EA, fo is the triangle CDE to the triangle ADE ; therefore the triangle BDE is to the triangle ADE, as the triangle CDE to the triangle ADE ; that is, the triangles BDE, CDE have the fame ratio to the
f . 5. triangle ADE ; and therefore ᶠ the triangle BDE is equal to the triangle CDE : And they are on the fame bafe DE ; but e-
g 39. 1. qual triangles on the fame bafe are between the fame parallels ᵍ ; therefore DE is parallel to BC. Wherefore, if a ftraight line, &c. Q. E. D.

PROP. III. THEOR.

IF the angle of a triangle be divided into two equal angles, by a ftraight line which alfo cuts the bafe ; the fegments of the bafe fhall have the fame ratio which the other fides of the triangle have to one another : And if the fegments of the bafe have the fame ratio which the other fides of the triangle have to one another, the ftraight line drawn from the vertex to the point of fection, divides the vertical angle into two equal angles.

Let the angle BAC of any triangle ABC be divided into two equal angles by the ftraight line AD: BD is to DC, as BA to AC.

Through

Through the point C draw CE parallel [a] to DA, and let BA produced meet CE in E. Becaufe the ftraight line AC meets the parallels AD, EC, the angle ACE is equal to the alternate [a] angle CAD [b]: But CAD, by the hypothefis, is equal to the [b] 29. 1. angle BAD; wherefore BAD is equal to the angle ACE. A-gain, becaufe the ftraight line BAE meets the parallels AD, EC, the outward angle BAD is equal to the inward and op-pofite angle AEC: But the angle ACE has been proved e-qual to the angle BAD; there-fore alfo ACE is equal to the angle AEC, and confequently the fide AE is equal to the fide [c] AC: And becaufe AD is drawn parallel to one of the [c] 6. 1. fides of the triangle BCE, viz. to EC, BD is to DC, as BA to AE [d]; but AE is equal to AC; therefore, as BD to DC, fo is [d] 2. 6. BA to AC [e]. [e] 7. 5.
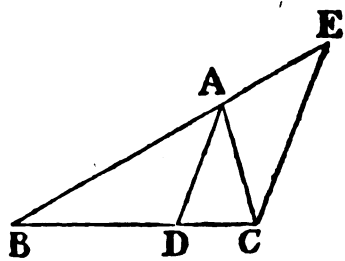
Let now BD be to DC, as BA to AC, and join AD; the angle BAC is divided into two equal angles by the ftraight line AD.

The fame conftruction being made; becaufe, as BD to DC, fo is BA to AC; and as BD to DC, fo is BA to AE [d], becaufe AD is parallel to EC; therefore BA is to AC, as BA to AE [f]: [f] 11. 5. Confequently AC is equal to AE [g], and the angle AEC is there- [g] 9. 5. fore equal to the angle ACE [h]: But the angle AEC is equal to h [h] 5. 1. the outward and oppofite angle BAD; and the angle ACE is equal to the alternate angle CAD [b]: Wherefore alfo the angle BAD is equal to the angle CAD: Therefore the angle BAC is cut into two equal angles by the ftraight line AD. Therefore, if the angle, &c. Q. E. D.

PROP. A. THEOR.

IF the outward angle of a triangle made by producing
one of its fides, be divided into two equal angles, by
a ftraight line which alfo cuts the bafe produced ; the
fegments between the dividing line and the extremities
of the bafe have the fame ratio which the other fides of
the triangle have to one another : And if the fegments
of the bafe produced, have the fame ratio which the o-
ther fides of the triangle have, the ftraight line drawn
from the vertex to the point of fection divides the out-
ward angle of the triangle into two equal angles.

Let the outward angle CAE of any triangle ABC be divided
into two equal angles by the ftraight line AD which meets the
bafe produced in D : BD is to DC, as BA to AC.

a 31. 1.   Through C draw CF parallel to AD^a; and becaufe the ftraight
line AC meets the parallels AD, FC, the angle ACF is equal
b 29. 1. to the alternate angle CAD^b: But CAD is equal to the angle
c Hyp.   DAE^c; therefore alfo DAE is equal to the angle ACF. Again,
becaufe the ftraight line FAE meets the parallels AD, FC, the
outward angle DAE is e-
qual to the inward and op-
pofite angle CFA: But the
angle ACF has been proved
equal to the angle DAE ;
therefore alfo the angle
ACF is equal to the angle
CFA, and confequently the
fide AF is equal to the fide
d 6. 1.   AC^d: And becaufe AD is parallel to FC, a fide of the triangle
e 2. 6.   BCF, BD is to DC, as BA to AF^e; but AF is equal to AC ;
therefore BD is to DC, fo is BA to AC.
    Let now BD be to DC, as BA to AC, and join AD ; the
angle CAD is equal to the angle DAE.
    The fame conftruction being made, becaufe BD is to DC,
f 11. 5. as BA to AC ; and that BD is alfo to DC, as BA to AF^e ;
g 9. 5. therefore BA is to AC, as BA to AF^f ; wherefore AC is equal
h 5. 1. to AF^g, and the angle AFC equal^h to the angle ACF: But
the
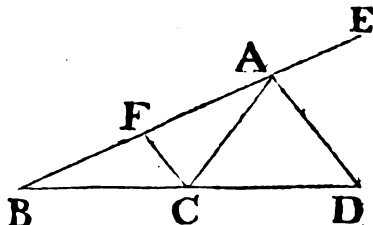
the angle AFC is equal to the outward angle EAD, and the
angle ACF to the alternate angle CAD; therefore alfo EAD is
equal to the angle CAD. Wherefore, if the outward, &c.
Q. E. D.

PROP. IV. THEOR.

THE fides about the equal angles of equiangular tri-
angles are proportionals; and thofe which are op-
pofite to the equal angles are homologous fides, that is,
are the antecedents or confequents of the ratios.

Let ABC, DCE be equiangular triangles, having the angle
ABC equal to the angle DCE, and the angle ACB to the angle
DEC, and confequently ª the angle BAC equal to the angle ª 32. 1.
CDE. The fides about the equal angles of the triangles ABC,
DCE are proportionals; and thofe are the homologous fides
which are oppofite to the equal angles.

Let the triangle DCE be placed, fo that its fide CE may be
contiguous to BC, and in the fame ftraight line with it: And
becaufe the angles ABC, ACB are together lefs than two right
angles ᵇ, ABC and DEC, which is ⸬⸬   b 17. 1.
equal to ACB, are alfo lefs than
two right angles; wherefore BA,   c 12. Ax. 1.
ED produced fhall meet ᶜ; let them
be produced and meet in the point
F: And becaufe the angle ABC is
equal to the angle DCE, BF is pa-
rallel ᵈ to CD. Again, becaufe the   d 28. 1.
angle ACB is equal to the angle
DEC, AC is parallel to FE ᵈ:
Therefore FACD is a parallelogram; and confequently AF is
equal to CD, and AC to FD ᵉ: And becaufe AC is parallel to ᵉ 34. 1.
FE, one of the fides of the triangle FBE, BA is to AF, as BC
to CE ᶠ: But AF is equal to CD; therefore ᵍ, as BA to CD, fo ᶠ 2. 6.
is BC to CE; and alternately, as AB to BC, fo is DC to CE ᵇ: ᵍ 7. 5.
Again, becaufe CD is parallel to BF, as BC to CE, fo is FD to
DE ᶠ; but FD is equal to AC; therefore, as BC to CE, fo is AC
to DE: And alternately, as BC to CA, fo CE to ED: Therefore,
becaufe it has been proved that AB is to BC, as DC to CE,
and as BC to CA, fo CE to ED, ex aequali ʰ, BA is to AC, as ʰ 22. 5.
CD to DE. Therefore the fides, &c. Q. E. D.
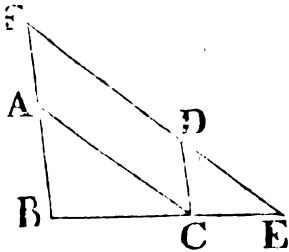
L   PROP.

PROP. V. THEOR.

IF the fides of two triangles, about each of their angles, be proportionals, the triangles fhall. be equiangular, and have their equal angles oppofite to the homologous fides.

Let the triangles ABC, DEF have their fides proportionals, fo that AB is to BC, as DE to EF; and BC to CA, as EF to FD; and confequently, ex aequali, BA to AC, as ED to DF; the triangle ABC is equiangular to the triangle DEF, and their equal angles are oppofite to the homologous fides, viz. the angle ABC equal to the angle DEF, and BCA to EFD, and alfo BAC to EDF.

a 23. I.   At the points E, F, in the ftraight line EF, make a the angle FEG equal to the angle ABC, and the angle EFG equal to BCA ; wherefore the remain-
ing angle BAC is equal to
b 32. I.   the remaining angle EGF b, and the triangle ABC is therefore equiangular to the triangle GEF; and confe-quently they have their fides oppofite to the equal angles
c 4. 6.   proportionals c : Wherefore, as AB to BC, fo is GE to EF;
d II. 5.   but as AB to BC, fo is DE to EF; therefore as DE to EF, fo d GE to EF: Therefore DE and GE have the fame ratio to EF,
e 9. 5.   and confequently are equal e : For the fame reafon, DF is equal to FG : And becaufe, in the triangles DEF, GEF, DE is equal to EG, and EF common, the two fides DE, EF are equal to the two GE, EF, and the bafe DF is equal to the bafe GF ;
f 8. I.   therefore the angle DEF is equal f to the angle GEF, and the other angles to the other angles which are fubtended by the e-
g 4. I.   qual fides g. Wherefore the angle DFE is equal to the angle GFE, and EDF to EGF: And becaufe the angle DEF is equal to the angle GEF, and GEF to the angle ABC ; therefore the angle ABC is equal to the angle DEF: For the fame reafon, the angle ACB is equal to the angle DFE, and the angle at A to the angle at D.   Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if the fides, &c. Q. E. D.
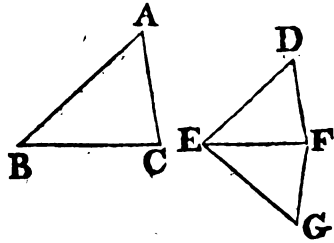
PROP.

PROP. VI. THEOR.

IF two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are oppofite to the homologous sides.

Let the triangles ABC, DEF have the angle BAC in the one equal to the angle EDF in the other, and the sides about those angles proportionals; that is, BA to AC, as ED to DF; the triangles ABC, DEF are equiangular, and have the angle ABC equal to the angle DEF, and ACB to DFE.

At the points D, F, in the ftraight line DF, make ª the ª 23. 1. angle FDG equal to either of the angles BAC, EDF; and the angle DFG equal to the angle ACB: Wherefore the remaining angle at B is equal to the remaining one at G ᵇ, and confe- b 32. 1. quently the triangle ABC is equiangular to the tri- angle DGF; and there- fore as BA to AC, fo is ᶜ c 4. 6. GD to DF: But, by the hypothefis, as BA to AC,
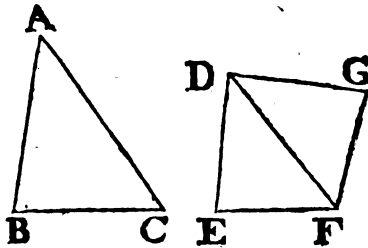
fo is ED to DF; as therefore ED to DF, fo is ᵈ GD to DF; d 11. 5. wherefore ED is equal ᵉ to DG; and DF is common to the two e 9. 5. triangles EDF, GDF: Therefore the two fides ED, DF are e- qual to the two fides GD, DF; and the angle EDF is equal to the angle GDF; wherefore the bafe EF is equal to the bafe FG ᶠ, f 4. 1. and the triangle EDF to the triangle GDF, and the remaining angles to the remaining angles, each to each, which are fub- tended by the equal fides: Therefore the angle DFG is equal to the angle DFE, and the angle at G to the angle at E: But the angle DFG is equal to the angle ACB; therefore the angle ACB is equal to the angle DFE: And the angle BAC is equal to the angle EDF ᵍ; wherefore alfo the remaining angle at B g Hyp. is equal to the remaining angle at E. Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if two triangles, &c. Q. E. D.

L 2   PROP.

P R O P. VII. T H E O R.

IF two triangles have one angle of the one equal to one angle of the other, and the fides about two other angles proportionals, then, if each of the remaining angles be either lefs, or not lefs, than a right angle; or if one of them be a right angle: The triangles fhall be equiangular, and have thofe angles equal about which the fides are proportionals.

Let the two triangles ABC, DEF have one angle in the one equal to one angle in the other, viz. the angle BAC to the angle EDF, and the fides about two other angles ABC, DEF proportionals, fo that AB is to BC, as DE to EF; and, in the firft cafe, let each of the remaining angles at C, F be lefs than a right angle. The triangle ABC is equiangular to the triangle DEF, viz. the angle ABC is equal to the angle DEF, and the remaining angle at C to the remaining angle at F.

For, if the angles ABC, DEF be not equal, one of them is greater than the other; let ABC be the greater, and at the point B, in the ftraight line AB, make the angle ABG equal to the angle *a* DEF: And becaufe the angle at A is equal to the angle at D, and the angle ABG to the angle DEF; the remaining angle AGB is equal *b* to the remaining angle DFE: Therefore the triangle ABG is equiangular to the triangle DEF; wherefore *c* as AB is to BG, fo is DE to EF; but as DE to EF, fo, by hypothefis, is AB to BC; therefore as AB to BC, fo is AB to BG *d*; and becaufe AB has the fame ratio to each of the lines BC, BG; BC is equal *e* to BG, and therefore the angle BGC is equal to the angle BCG *f*: But the angle BCG is, by hypothefis, lefs than a right angle; therefore alfo the angle BGC is lefs than a right angle, and the adjacent angle AGB muft be greater than a right angle *g*. But it was proved that the angle AGB is equal to the angle at F; therefore the angle at F is greater than a right angle: But, by the hypothefis, it is lefs than a right angle; which is abfurd. Therefore
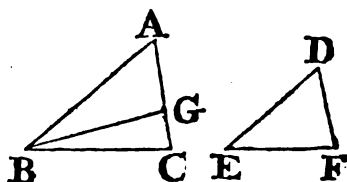
a 23. 1.
b 32. 1.
c 4. 6.
d 11. 5.
e 9. 5.
f 5. 1.
g 13. 1.

fore

fore the angles ABC, DEF are not unequal, that is, they are equal: And the angle at A is equal to the angle at D; wherefore the remaining angle at C is equal to the remaining angle at F: Therefore the triangle ABC is equiangular to the triangle DEF.

Next, Let each of the angles at C, F be not lefs than a right angle: The triangle ABC is alfo in this cafe equiangular to the triangle DEF.

The fame conftruction being made, it may be proved in like manner that BC is equal to BG, and the angle at C equal to the angle BGC: But the angle at C is not lefs than a right angle; therefore the angle BGC is not lefs than a right angle: Wherefore two angles of the triangle BGC are together not lefs than two right angles, which is impofsible [h]; and therefore the triangle ABC may be proved to be equiangular to the triangle DEF, as in the firft cafe.

h 17. 1.
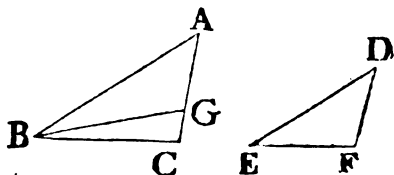
Laftly, Let one of the angles at C, F, viz. the angle at C, be a right angle; in this cafe likewife the triangle ABC is equiangular to the triangle DEF.

For, if they be not equiangular, make, at the point B of the ftraight line AB, the angle ABG equal to the angle DEF; then it may be proved, as in the firft cafe, that BG is equal to BC: But the angle BCG is a right angle, therefore [i] the angle BGC is alfo a right angle; whence two of the angles of the triangle BGC are together not lefs than two right angles, which is impofsible [h]: Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if two triangles, &c. Q. E. D.

i 5. 1.
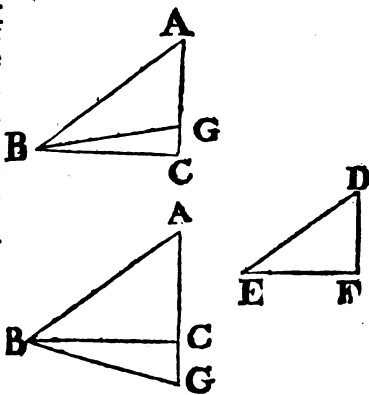
L 3

P R O P.

PROP. VIII. THEOR.

IN a right angled triangle, if a perpendicular be drawn from the right angle to the bafe; the triangles on each fide of it are fimilar to the whole triangle, and to one another.

Let ABC be a right angled triangle, having the right angle BAC; and from the point A let AD be drawn perpendicular to the bafe BC: The triangles ABD, ADC are fimilar to the whole triangle ABC, and to one another.

Becaufe the angle BAC is equal to the angle ADB, each of them being a right angle, and that the angle at B is common to the two triangles ABC, ABD; the remaining angle ACB is equal to the remaining angle BAD^a: Therefore the triangle ABC is equiangular to the triangle ABD, and the fides about their equal angles are pro-

a 32. 1.

b 4. 6.
c 1. Def. 6.

portionals^b; wherefore the triangles are fimilar^c: In the like
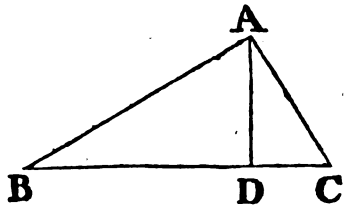
manner it may be demonftrated, that the triangle ADC is equiangular and fimilar to the triangle ABC: And the triangles ABD, ADC, being both equiangular and fimilar to ABC, are equiangular and fimilar to each other. Therefore, in a right angle, &c. Q. E. D.

COR. From this it is manifeft, that the perpendicular drawn from the right angle of a right angled triangle to the bafe, is a mean proportional between the fegments of the bafe: And alfo that each of the fides is a mean proportional between the bafe, and its fegment adjacent to that fide: Becaufe in the triangles BDA, ADC, BD is to DA, as DA to DC^b; and in the triangles ABC, DBA, BC is to BA, as BA to BD^b; and in the triangles ABC, ACD, BC is to CA, as CA to CD^b.

PROP.

PROP. IX. PROB.

FROM a given ftraight line to cut off any part re- See N. quired.

Let AB be the given ftraight line; it is required to cut off any part from it.

From the point A draw a ftraight line AC making any angle with AB; and in AC take any point D, and take AC the fame multiple of AD, that AB is of the part which is to be cut off from it; join BC, and draw DE parallel to it: Then AE is the part required to be cut off.
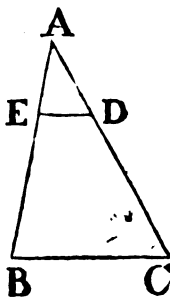
Becaufe ED is parallel to one of the fides of the triangle ABC, viz. to BC, as CD is to DA, fo is [a] BE to EA; and, by compofition [b], CA is to AD, as BA to AE: But CA is a multiple of AD; therefore [c] BA is the fame multiple of AE: Whatever part therefore AD is of AC, AE is the fame part of AB: Wherefore, from the ftraight line AB the part required is cut off. Which was to be done.

a 2. 6.
b 18. 5.
c D. 5.

PROP. X. PROB.

TO divide a given ftraight line fimilarly to a given divided ftraight line, that is, into parts that fhall have the fame ratios to one another which the parts of the divided given ftraight line have.

Let AB be the ftraight line given to be divided, and AC the divided line; it is required to divide AB fimilarly to AC.

Let AC be divided in the points D, E; and let AB, AC be placed fo as to contain any angle, and join BC, and through the points D, E, draw [a] DF, EG parallels to it; and through D draw DHK parallel to AB: Therefore each of the figures FH, HB, is a parallelogram; wherefore DH is equal [b] to FG, and b 34. 1.

a 31. 1.

L 4   HK

HK to GB: And because HE is parallel to KC, one of the sides of the triangle DKC, as CE to ED, so is
ᶜ KH to HD: But KH is equal to BG, and HD to GF; therefore, as CE to ED, so is BG to GF: Again, because FD is parallel to EG, one of the sides of the triangle AGE, as ED to DA, so is GF to FA: But it has been proved that CE is to ED, as BG to GF; and as ED to DA, so GF to FA: Therefore the given straight line AB is divided similarly to AC. Which was to be done.
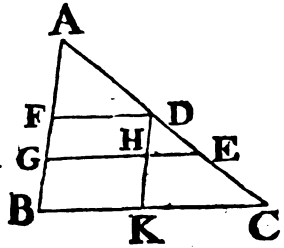
P R O P. XI.   P R O B.

TO find a third proportional to two given straight lines.

Let AB, AC be the two given straight lines, and let them be placed so as to contain any angle; it is required to find a third proportional to AB, AC.

Produce AB, AC to the points D, E; and make BD equal to AC; and having joined BC, through D, draw DE parallel to it ᵃ.
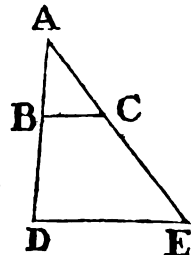
Because BC is parallel to DE, a side of the triangle ADE, AB is ᵇ to BD, as AC to CE: But BD is equal to AC; as therefore AB to AC, so is AC to CE. Wherefore to the two given straight lines AB, AC a third proportional CE is found. Which was to be done.

P R O P. XII.   P R O B.

TO find a fourth proportional to three given straight lines.

Let A, B, C be the three given straight lines; it is required to find a fourth proportional to A, B, C.

Take

Take two ſtraight lines DE, DF, containing any angle EDF; **Book VI.**
and upon theſe make DG
equal to A, GE equal to B,
and DH equal to C; and
having joined GH, draw EF
parallel [a] to it through the
point E : And becauſe GH
is parallel to EF, one of the
ſides of the triangle DEF,
DG is to GE, as DH to
HF [b]; but DG is equal to
A, GE to B, and DH to C;
therefore, as A is to B, ſo is C to HF. Wherefore to the
three given ſtraight lines, A, B, C a fourth proportional HF is
found. Which was to be done.
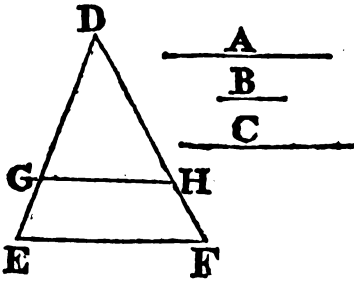
a 31. 1.

b 2. 6.

PROP. XIII. PROB.

TO find a mean proportional between two given
ſtraight lines.

Let AB, BC be the two given ſtraight lines; it is required
to find a mean proportional between them.
Place AB, BC in a ſtraight line, and upon AC deſcribe the
ſemicircle ADC, and from the
point B draw [a] BD at right an-
gles to AC, and join AD, DC.
Becauſe the angle ADC in a
ſemicircle is a right angle [b], and
becauſe in the right angled tri-
angle ADC, DB is drawn from
the right angle perpendicular to
the baſe, DB is a mean propor-
tional between AB, BC the ſegments of the baſe [c]: Therefore
between the two given ſtraight lines AB, BC, a mean propor-
tional DB is found. Which was to be done.

a 11. 1.

b 31. 3.

c Cor. 8. 6.
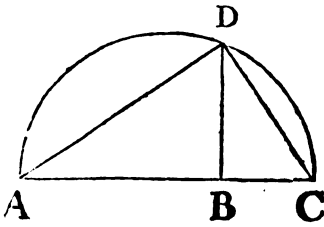
PROP.

PROP. XIV. THEOR.

EQUAL parallelograms which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: And parallelograms that have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let AB, BC be equal parallelograms, which have the angles at B equal, and let the sides DB, BE be placed in the same
a 14. 1. straight line; wherefore also FB, BG are in one straight line [a] : The sides of the parallelograms AB, BC about the equal angles, are reciprocally proportional; that is, DB is to BE, as GB to BF.

Complete the parallelogram FE; and because the parallelogram AB is equal to BC, and that FE is another parallelogram, AB is to FE, as BC to
b 7. 5. FE [b] : But as AB to FE, so is
c 1. 6. the base DB to BE [c]; and, as BC to FE, so is the base GB to BF; therefore, as DB to BE,
d 11. 5. so is GB to BF [d]. Wherefore the sides of the parallelograms AB, BC about their equal angles are reciprocally proportional.
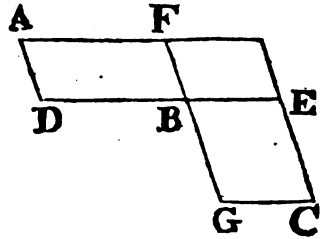
But, let the sides about the equal angles be reciprocally proportional, viz. as DB to BE, so GB to BF; the parallelogram AB is equal to the parallelogram BC.

Because, as DB is to BE, so GB to BF; and as DB to BE, so is the parallelogram AB to the parallelogram FE; and as GB to BF, so is the parallelogram BC to the parallelogram FE;
therefore as AB to FE, so BC to FE [d] : Wherefore the parallelo-
e 9. 5. gram AB is equal [e] to the parallelogram BC. Therefore equal parallelograms, &c. Q. E. D.

PROP.

PROP. XV. THEOR.

EQUAL triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional : And triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let ABC, ADE be equal triangles, which have the angle BAC equal to the angle DAE; the sides about the equal angles of the triangles are reciprocally proportional; that is, CA is to AD, as EA to AB.

Let the triangles be placed so that their sides CA, AD be in one straight line; wherefore also EA and AB are in one straight line [a]; and join BD. Because the triangle ABC is e- [a] 14. 1. qual to the triangle ADE, and that ABD is another triangle; therefore as the triangle CAB is to the triangle BAD, so is triangle EAD to triangle DAB [b] : But as triangle [b] 7. 5. CAB to triangle BAD, so is the base CA to AD [c]; and as triangle [c] 1. 6. EAD to triangle DAB, so is the base EA to AB [c]; as therefore CA to AD, so is EA to AB [d]: [d] 11. 5. wherefore the sides of the triangles ABC, ADE about the equal angles are reciprocally proportional.
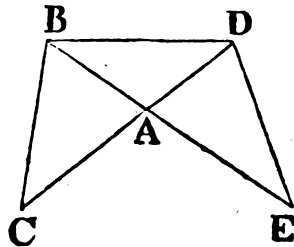
But let the sides of the triangles ABC, ADE about the equal angles be reciprocally proportional, viz. CA to AD, as EA to AB; the triangle ABC is equal to the triangle ADE.

Having joined BD as before ; because, as CA to AD, so is EA to AB ; and as CA to AD, so is triangle ABC to triangle BAD [c]; and as EA to AB, so is triangle EAD to triangle BAD [c]; therefore [d] as triangle BAC to triangle BAD, so is triangle EAD to triangle BAD; that is, the triangles BAC, EAD have the same ratio to the triangle BAD: Wherefore the triangle ABC is equal [e] to the triangle ADE. Therefore equal [e] 9. 5. triangles, &c. Q. E. D.

PROP.

PROP. XVI. THEOR.

IF four ftraight lines be proportionals, the rectangle
contained by the extremes is equal to the rectangle
contained by the means : And if the rectangle contained
by the extremes be equal to the rectangle contained by
the means, the four ftraight lines are proportionals.

Let the four ftraight lines, AB, CD, E, F be proportionals,
viz. as AB to CD, fo E to F; the rectangle contained by AB,
F is equal to the rectangle contained by CD, E.

From the points A, C draw ᵃ AG, CH at right angles to
AB, CD ; and make AG equal to F, and CH equal to E, and
complete the parallelograms BG, DH : Becaufe, as AB to CD,
fo is E to F; and that E is equal to CH, and F to AG ; AB
is ᵇ to CD, as CH to AG : Therefore the fides of the parallelo-
grams BG, DH about the equal angles are reciprocally pro-
portional ; but parallelograms which have their fides about e-
qual angles reciprocally proportional, are equal to one another ᶜ ;
therefore the parallelogram BG is equal to the parallelogram
DH : And the parallelogram
BG is contained by the
ftraight lines AB, F; becaufe
AG is equal to F ; and the
parallelogram DH is con-
tained by CD and E; becaufe
CH is equal to E : There-
fore the rectangle contained
by the ftraight lines AB, F
is equal to that which is
contained by CD and E.
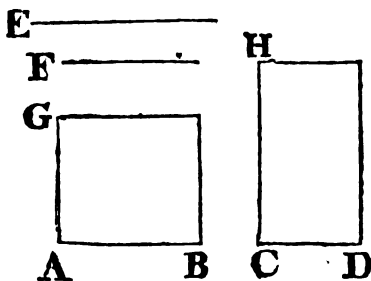
And if the rectangle contained by the ftraight lines AB, F
be equal to that which is contained by CD, E ; thefe four lines
are proportionals, viz. AB is to CD, as E to F.

The fame conftruction being made, becaufe the rectangle
contained by the ftraight lines AB, F is equal to that which is
contained by CD, E, and that the rectangle BG is contained
by AB, F, becaufe AG is equal to F ; and the rectangle DH
by CD, E, becaufe CH is equal to E ; therefore the parallelo-
gram BG is equal to the parallelogram DH ; and they are e-
quiangular:

quiangular: But the fides about the equal angles of equal pa-
rallelograms are reciprocally proportional c : Wherefore, as AB
to CD, fo is CH to AG; and CH is equal to E, and AG to F : c 14. 6.
As therefore AB is to CD, fo E to F. Wherefore, if four, &c.
Q. E. D.

PROP. XVII. THEOR.

IF three ſtraight lines be proportionals, the rectangle
contained by the extremes is equal to the ſquare of
the mean: And if the rectangle contained by the ex-
tremes be equal to the ſquare of the mean, the three
ſtraight lines are proportionals.

Let the three ſtraight lines A, B, C be proportionals, viz.
as A to B, fo B to C; the rectangle contained by A, C is equal
to the ſquare of B.

Take D equal to B; and becauſe as A to B, fo B to C, and
that B is equal to D; A is ᵃ to B, as D to C: But if four ſtraight ᵃ 7. 5.
lines be proportionals,
the rectangle contain-
ed by the extremes is
equal to that which
is contained by the
means ᵇ: Therefore the
rectangle contained by
A, C is equal to that
contained by B, D:
But the rectangle con-
tained by B, D is the
ſquare of B; becauſe B is equal to D: Therefore the rectangle
contained by A, C is equal to the ſquare of B.
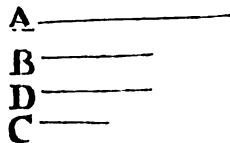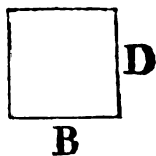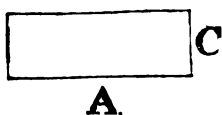
b 16. 6.

And if the rectangle contained by A, C be equal to the
ſquare of B; A is to B, as B to C.

The ſame conſtruction being made, becauſe the rectangle
contained by A, C is equal to the ſquare of B, and the ſquare
of B is equal to the rectangle contained by B, D, becauſe B is
equal to D; therefore the rectangle contained by A, C is equal
to that contained by B, D: But if the rectangle contained by
the extremes be equal to that contained by the means, the four
ſtraight lines are proportionals ᵇ: Therefore A is to B, as D to
C;

C ; but B is equal to D ; wherefore as A to B, so B to C. Therefore, if three ſtraight lines, &c. Q. E. D.

PROP. XVIII. THEOR.

UPON a given ſtraight line to deſcribe a rectilineal figure ſimilar, and ſimilarly ſituated to a given rectilineal figure.

Let AB be the given ſtraight line, and CDEF the given rectilineal figure of four ſides ; it is required upon the given ſtraight line AB to deſcribe a rectilineal figure ſimilar, and ſimilarly ſituated to CDEF.

Join DF, and at the points A, B in the ſtraight line AB, make ᵃ the angle BAG equal to the angle at C, and the angle ABG equal to the angle CDF ; therefore the remaining angle CFD is equal to the remaining angle AGB ᵇ : Wherefore the triangle FCD is e-

a 23. I.

b 32. I.

quiangular to the tri-
angle GAB : Again,
at the points G, B
in the ſtraight line
GB make ᵃ the angle
BGH equal to the
angle DFE, and the
angle GBH equal to
FDE ; therefore the
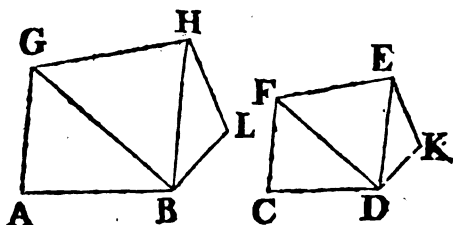
remaining angle FED is equal to the remaining angle GHB, and the triangle FDE equiangular to the triangle GBH : Then, becauſe the angle AGB is equal to the angle CFD, and BGH to DFE, the whole angle AGH is equal to the whole CFE : For the ſame reaſon, the angle ABH is equal to the angle CDE ; alſo the angle at A is equal to the angle at C, and the angle GHB to FED : Therefore the rectilineal figure ABHG is equiangular to CDEF : But likewiſe theſe figures have their ſides a-bout the equal angles proportionals : Becauſe the triangles GAB, FCD being equiangular, BA is ᶜ to AG, as DC to CF ; and becauſe AG is to GB, as CF to FD ; and as GB to GH, ſo, by reaſon of the equiangular triangles BGH, DFE, is FD to FE ; therefore, ex aequali ᵈ, AG is to GH, as CF to FE : In the ſame manner it may be proved that AB is to BH as CD to DE : And GH is to HB, as FE to ED ᵉ. Wherefore, becauſe
the

a 4. 6.

d 22. 5.

the rectilineal figures ABHG, CDEF are equiangular, and have their sides about the equal angles proportionals, they are similar to one another [e].

Next, Let it be required to describe upon a given straight line AB, a rectilineal figure similar, and similarly situated to the rectilineal figure CDKEF.

Join DE, and upon the given straight line AB describe the rectilineal figure ABHG similar, and similarly situated to the quadrilateral figure CDEF, by the former case; and at the points B, H in the straight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK; therefore the remaining angle at K is equal to the remaining angle at L: And because the figures ABHG, CDEF are similar, the angle GHB is equal to the angle FED, and BHL is equal to DEK; wherefore the whole angle GHL is equal to the whole angle FEK; For the same reason the angle ABL is equal to the angle CDK: Therefore the five sided figures AGHLB, CFEKD are equiangular: And because the figures AGHB, CFED are similar, GH is to HB, as FE to ED; and as HB to HL, so is ED to EK [c]; therefore, ex aequali [d], GH is to HL, as FE to EK: For the same reason, AB is to BL, as CD to DK: And BL is to LH, as [c] DK to KE, because the triangles BLH, DKE are equiangular: Therefore, because the five sided figures AGHLB, CFEKD are equiangular, and have their sides about the equal angles proportionals, they are similar to one another: And in the same manner a rectilineal figure of six or more sides may be described upon a given straight line similar to one given, and so on. Which was to be done.

PROP. XIX. THEOR.

SIMILAR triangles are to one another in the duplicate ratio of their homologous sides.

Let ABC, DEF be similar triangles, having the angle B equal to the angle E, and let AB be to BC, as DE to EF, so that the side BC is homologous to EF [a]: The triangle ABC has to the triangle DEF, the duplicate ratio of that which BC has to EF.

Take BG a third proportional to BC, EF [b], so that BC is to EF, as EF to BG, and join GA: Then, because as AB to BC, so DE to EF; alternately [c], AB is to DE, as BC to EF: But

Book VI.

d 11. 5.

as BC to EF, fo is EF to BG; therefore ^d as AB to DE, fo is EF to BG: Wherefore the fides of the triangles ABG, DEF which are about the equal angles, are reciprocally proportional: But triangles which have the fides about two equal angles reciprocally proportional are equal to one another ^e: Therefore the triangle ABG is equal to the triangle DEF: And becaufe as BC is to EF, fo EF to BG; and that if three ftraight lines be proportionals, the firft is

e 15. 6.

A
D
B G C E F

f 10. def. 5.

g 1. 6.
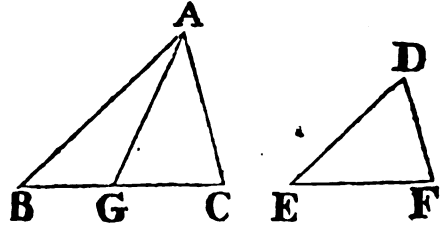
faid ^f to have to the third the duplicate ratio of that which it has to the fecond; BC therefore has to BG the duplicate ratio of that which BC has to EF: But as BC to BG, fo is ^g the triangle ABC to the triangle ABG. Therefore the triangle ABC has to the triangle ABG the duplicate ratio of that which BC has to EF: But the triangle ABG is equal to the triangle DEF; wherefore alfo the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF. Therefore fimilar triangles, &c. Q. E. D.

COR. From this it is manifeft, that if three ftraight lines be proportionals, as the firft is to the third, fo is any triangle upon the firft to a fimilar, and fimilarly defcribed triangle upon the fecond.

PROP. XX. THEOR.

SIMILAR polygons may be divided into the fame number of fimilar triangles, having the fame ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous fides have.

Let ABCDE, FGHKL be fimilar polygons, and let AB be the homologous fide to FG: The polygons ABCDE, FGHKL may be divided into the fame number of fimilar triangles, whereof each to each has the fame ratio which the polygons have; and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the fide AB has to the fide FG.

Join BE, EC, GL, LH: And becaufe the polygon ABCDE is

fimilar

similar to the polygon FGHKL, the angle BAE is equal to the angle GFL [a], and BA is to AE, as GF to FL [a] : Wherefore, because the triangles ABE, FGL have an angle in one equal to an angle in the other, and their sides about these equal angles proportionals, the triangle ABE is equiangular [b], and therefore similar to the triangle FGL [c]; wherefore the angle ABE is equal to the angle FGL : And, because the polygons are similar, the whole angle ABC is equal [a] to the whole angle FGH ; therefore the remaining angle EBC is equal to the remaining angle LGH : And because the triangles ABE, FGL are similar, EB is to BA, as LG to GF [c]; and also, because the polygons are similar, AB is to BC, as FG to GH [a]; therefore, ex aequali [d], EB is to BC, as LG to GH ; that is, the sides about the equal angles EBC, LGH are proportionals; therefore [b] the triangle EBC is equiangular to the triangle LGH, and similar to it [c].

For the same reason, the triangle ECD likewise is similar to the triangle LHK: Therefore the similar polygons ABCDE,

FGHKL are divided into the same number of similar triangles.
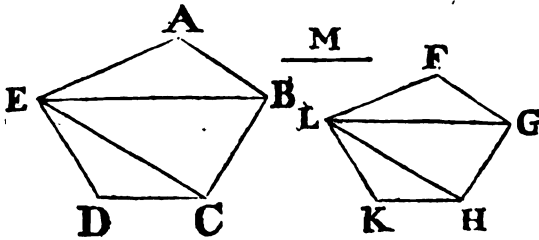
Also these triangles have, each to each, the same ratio which the polygons have to one another, the antecedents being ABE, EBC, ECD, and the consequents FGL, LGH, LHK : And the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the side AB has to the homologous side FG.

Because the triangle ABE is similar to the triangle FGL, ABE has to FGL the duplicate ratio [e] of that which the side BE has to the side GL : For the same reason, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL : Therefore, as the triangle ABE to the triangle FGL, so [f] is the triangle BEC to the triangle GLH. Again, because the triangle EBC is similar to the triangle LGH, EBC has to LGH the duplicate ratio of that which the side EC has to the side LH : For the same reason, the triangle ECD has to the triangle

M   LHK,

LHK, the duplicate ratio of that which EC has to LH: **As**

therefore the triangle EBC to the triangle LGH, fo is f the
triangle ECD to the triangle LHK: But it has been proved that
the triangle EBC is likewife to the triangle LGH, as the triangle
ABE to the triangle FGL. Therefore, as the triangle ABE is
to the triangle FGL, fo is triangle EBC to triangle LGH, and
triangle ECD to triangle LHK: And therefore, as one of the
antecedents to one of the confequents, fo are all the antecedents

to all the confequents g. Wherefore, as the triangle ABE to the
triangle FGL, fo is the polygon ABCDE to the polygon
FGHKL: But the triangle ABE has to the triangle FGL, the
duplicate ratio of that which the fide AB has to the homologous
fide FG. Therefore alfo the polygon ABCDE has to the poly-
gon FGHKL the duplicate ratio of that which AB has to the
homologous fide FG. Wherefore fimilar polygons, &c. Q. E. D.

Cor. 1. In like manner, it may be proved, that fimilar four
fided figures, or of any number of fides, are one to another in
the duplicate ratio of their homologous fides, and it has already
been proved in triangles. Therefore, univerfally fimilar recti-
lineal figures are to one another in the duplicate ratio of their
homologous fides.

Cor. 2. And if to AB, FG, two of the homologous fides,

a third proportional M be taken, AB has h to M the duplicate
ratio of that which AB has to FG: But the four fided figure or
polygon upon AB has to the four fided figure or polygon upon
FG likewife the duplicate ratio of that which AB has to FG:
Therefore, as AB is to M, fo is the figure upon AB to the figure

upon FG, which was alfo proved in triangles i. Therefore,
univerfally, it is manifeft, that if three ftraight lines be propor-
tionals, as the firft is to the third, fo is any rectilineal figure up-
on the firft, to a fimilar and fimilarly defcribed rectilineal fi-
gure upon the fecond.

P R O P.

PROP. XXI. THEOR.

RECTILINEAL figures which are fimilar to the fame rectilineal figure, are alfo fimilar to one another.

Let each of the rectilineal figures A, B be fimilar to the rectilineal figure C: The figure A is fimilar to the figure B.

Becaufe A is fimilar to C, they are equiangular, and alfo have their fides about the equal angles proportionals [a]. Again, **a** 1. def. 6;
becaufe B is fimilar to C, they are equiangular, and have their fides about the equal angles proportionals [a]: There- fore the figures A, B are each of them equi-

angular to C, and have the fides about the equal angles of each of them and of C proportionals. Wherefore the rectilineal fi- gures A and B are equiangular [b], and have their fides about the **b** 1. Ax. 1; equal angles proportionals [c]. Therefore A is fimilar [a] to B. **c** 11. 5. Q. E. D.
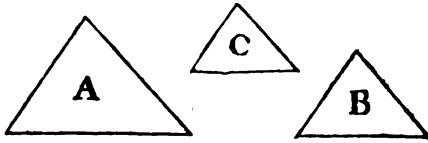
PROP. XXII. THEOR.

IF four ftraight lines be proportionals, the fimilar rec- tilineal figures fimilarly defcribed upon them fhall alfo be proportionals; and if the fimilar rectilineal figures fimilarly defcribed upon four ftraight lines be proportion- als, thofe ftraight lines fhall be proportionals.

Let the four ftraight lines AB, CD, EF, GH be propor- tionals, viz AB to CD, as EF to GH, and upon AB, CD let the fimilar rectilineal figures KAB, LCD be fimilarly defcri- bed; and upon EF, GH the fimilar rectilineal figures MF, NH, in like manner: The rectilineal figure KAB is to LCD, as MF to NH.

To AB, CD take a third proportional [a] X; and to EF, GH **a** 11. 6. a third proportional O: And becaufe AB is to CD, as EF to GH, and that CD is [b] to X, as GH to O; wherefore, ex ae- **b** 11. 5. quali [c], as AB to X, fo EF to O: But as AB to X, fo is [d] the **c** 22. 5.
d 2. Cor.
rectilineal 20. 6.

M 2

rectilineal KAB to the rectilineal LCD, and as EF to O, fo is
d the rectilineal MF to the rectilineal NH: Therefore, as KAB
to LCD, fo b is MF to NH.

And if the rectilineal KAB be to LCD, as MF to NH; the
ftraight line AB is to CD, as EF to GH.

Make e as AB to CD, fo EF to PR, and upon PR defcribe f
the rectilineal figure SR fimilar and fimilarly fituated to either
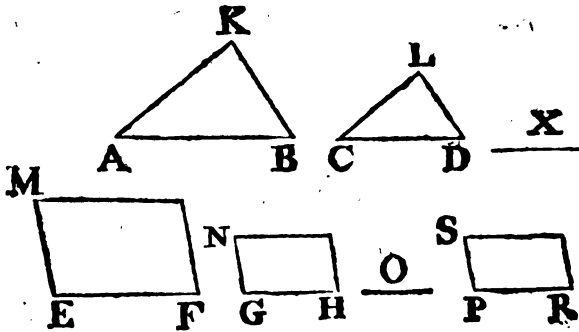
of the figures MF, NH: Then, becaufe as AB to CD, fo is EF
to PR, and that upon AB, CD are defcribed the fimilar and
fimilarly fituated rectilineals KAB, LCD, and upon EF, PR,
in like manner, the fimilar rectilineals MF, SR; KAB is to
LCD, as MF to SR; but, by the hypothefis, KAB is to LCD,
as MF to NH; and therefore the rectilineal MF having the
fame ratio to each of the two NH, SR, thefe are equal g to one
another: They are alfo fimilar, and fimilarly fituated; there-
fore GH is equal to PR: And becaufe as AB to CD, fo is EF
to PR, and that PR is equal to GH; AB is to CD, as EF to
GH.   If therefore four ftraight lines, &c. Q. E. D.

PROP. XXIII.   THEOR.

EQUIANGULAR parallelograms have to one another
the ratio which is compounded of the ratios of
their fides.

Let AC, CF be equiangular parallelograms, having the angle
BCD equal to the angle ECG: The ratio of the parallelogram
AC to the parallelogram CF, is the fame with the ratio which
is compounded of the ratios of their fides.

Let

Let BC, CG be placed in a ſtraight line; therefore DC and **Book VI.**
CE are alſo in a ſtraight line ᵃ; and complete the parallelogram
DG; and, taking any ſtraight line K, make ᵇ as BC to CG, ᵃ 14. 1.
ſo K to L; and as DC to CE, ſo make ᵇ L to M: Therefore ᵇ 12. 6.
the ratios of K to L, and L to M, are the ſame with the ratios
of the ſides, viz. of BC to CG, and DC to CE. But the ra-
tio of K to M is that which is ſaid to be compounded ᶜ of the ᶜ A. def. 5.
ratios of K to L, and L to M: Wherefore alſo K has to M, the
ratio compounded of the ratios
of the ſides: And becauſe as BC
to CG, ſo is the parallelogram
AC to the parallelogram CH ᵈ; ᵈ 1. 6.
but as BC to CG, ſo is K to L;
therefore K is ᵉ to L, as the pa- ᵉ 11. 5.
rallelogram AC to the parallelo-
gram CH: Again, becauſe as DC
to CE, ſo is the parallelogram
CH to the parallelogram CF; but
as DC to CE, ſo is L to M;
wherefore L is ᵉ to M, as the pa-
rallelogram CH to the parallelogram CF: Therefore, ſince it
has been proved, that as K to L, ſo is the parallelogram AC to
the parallelogram CH; and as L to M, ſo the parallelogram CH
to the parrallelogram CF; ex aequali ᶠ, K to M, as the paral- ᶠ 22. 5.
lelogram AC to the parallelogram CF: But K has to M the
ratio which is compounded of the ratios of the ſides; therefore
alſo the parallelogram AC has to the parallelogram CF the ratio
which is compounded of the ratios of the ſides. Wherefore
equiangular parallelograms, &c. Q. E. D.
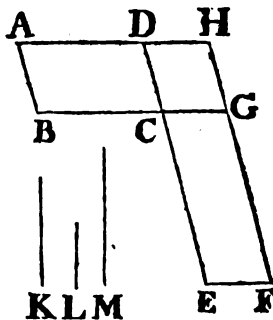
PROP. XXIV. THEOR.

THE parallelograms about the diameter of any pa- See N.
rallelogram, are ſimilar to the whole, and to one
another.

Let ABCD be a parallelogram, of which the diameter is
AC; and EG, HK the parallelograms about the diameter: The
parallelograms EG, HK are ſimilar both to the whole paralle-
logram ABCD, and to one another.
Becauſe DC, GF are parallels, the angle ADC is equal ᵃ to ᵃ 29. 1.
the angle AGF: For the ſame reaſon, becauſe BC, EF are pa-
rallels,

M 3

rallels, the angle ABC is equal to the angle AEF: And each
of the angles BCD, EFG is equal to the oppofite angle DAB[b],
and therefore are equal to one another; wherefore the paral-
lelograms ABCD, AEFG are equiangular: And becaufe the
angle ABC is equal to the angle AEF, and the angle BAC
common to the two triangles BAC, EAF, they are equiangu-
lar to one another; therefore[c] as AB
to BC, fo is AE to EF: And becaufe
the oppofite fides of parallelograms
are equal to one another[b], AB is[d] to
AD, as AE to AG; and DC to CB,
as GF to FE; and alfo CD to DA,
as FG to GA: Therefore the fides of
the parallelograms ABCD, AEFG a-
bout the equal angles are proportion-
als; and they are therefore fimilar to
one another[e]: For the fame reafon, the parallelogram ABCD
is fimilar to the parallelogram FHCK. Wherefore each of the
parallelograms GE, KH is fimilar to DB: But rectilineal figures
which are fimilar to the fame rectilineal figure, are alfo fimilar
to one another[f]; therefore the parallelogram GE is fimilar to
KH. Wherefore the parallelograms, &c. Q. E. D.
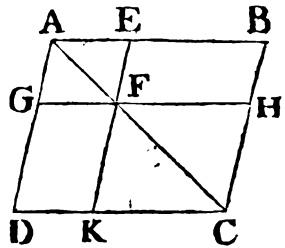
PROP. XXV. PROB.

TO defcribe a rectilineal figure which fhall be fimilar
to one, and equal to another given rectilineal fi-
gure.

Let ABC be the given rectilineal figure, to which the figure
to be defcribed is required to be fimilar, and D that to which
it muft be equal. It is required to defcribe a rectilineal figure
fimilar to ABC, and equal to D.
Upon the ftraight line BC defcribe[a] the parallelogram BE
equal to the figure ABC; alfo upon CE defcribe[a] the paralle-
logram CM equal to D, and having the angle FCE equal
to the angle CBL: Therefore BC and CF are in a ftraight
line[b], as alfo LE and EM: Between BC and CF find[c] a mean
proportional GH, and upon GH defcribe[d] the rectilineal fi-
gure KGH fimilar and fimilarly fituated to the figure ABC:
And becaufe BC is to GH as GH to CF, and if three ftraight
lines be proportionals, as the firft is to the third, fo is[e] the
                                                                      figure

figure upon the firſt to the ſimilar and ſimilarly deſcribed fi-
gure upon the ſecond; therefore as BC to CF, ſo is the recti-
lineal figure ABC to KGH: But as BC to CF, ſo is f the pa- f 1. 6.
rallelogram BE to the parallelogram EF: Therefore as the rec-
tilineal figure ABC is to KGH, ſo is the parallelogram BE to
the parallelogram EF⁸: And the rectilineal figure ABC is equal ⁸ 11. 5.
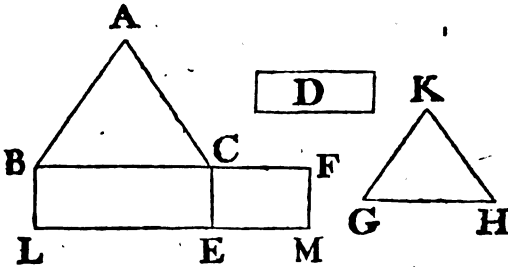
to the parallelogram BE; therefore the rectilineal figure KGH
is equal ᵇ to the parallelogram EF: But EF is equal to the fi- h 14. 5.
gure D; wherefore alſo KGH is equal to D; and it is ſimilar
to ABC. Therefore the rectilineal figure KGH has been de-
ſcribed ſimilar to the figure ABC, and equal to D. Which
was to be done.

PROP. XXVI. THEOR.

IF two ſimilar parallelograms have a common angle, and
be ſimilarly ſituated; they are about the ſame diameter.

Let the parallelograms ABCD, AEFG be ſimilar and ſimi-
larly ſituated, and have the angle DAB common. ABCD and
AEFG are about the ſame diameter.

For, if not, let, if poſſible, the
parallelogram BD have its dia-
meter AHC in a different ſtraight
line from AF the diameter of the
parallelogram EG, and let GF
meet AHC in H; and through
H draw HK parallel to AD or
BC: Therefore the parallelograms
ABCD, AKHG being about the
ſame diameter, they are ſimilar
to one another ᵃ: Wherefore as DA to AB, ſo is ᵇ GA to AK: ᵃ 24. 6.
But becauſe ABCD and AEFG are ſimilar parallelograms, ᵇ 1. def.
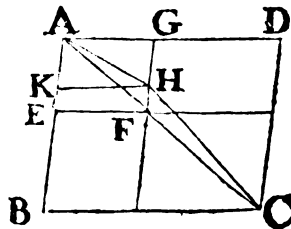
M 4   as

as DA is to AB, so is GA to AE; therefore c as GA to AE, to GA to AK; wherefore GA has the same ratio to each of the straight lines AE, AK; and consequently AK is equal d to AE, the less to the greater, which is impossible: Therefore ABCD and AKHG are not about the same diameter; wherefore ABCD and AEFG must be about the same diameter. Therefore, if two similar, &c. Q. E. D.

‘ To underftand the three following propofitions more eaſily, ‘ it is to be obſerved,

‘ 1. That a parallelogram is faid to be applied to a ftraight ‘ line, when it is defcribed upon it as one of its fides. 'Ex. gr. ‘ the parallelogram AC is faid to be applied to the ftraight line ‘ AB.

‘ 2. But a parallelogram AE is faid to be applied to a ftraight ‘ line AB, deficient by a parallelogram, when AD the bafe of ‘ AE is lefs than AB, and there-
‘ fore AE is lefs than the paral-
‘ lelogram AC defcribed upon
‘ AB in the fame angle, and be-
‘ tween the fame parallels, by
‘ the parallelogram DC; and DC
‘ is therefore called the defect of
‘ AE.
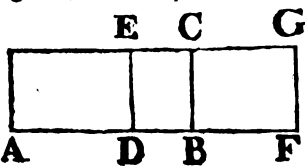
‘ 3. And a parallelogram AG is faid to be applied to a ftraight ‘ line AB, exceeding by a parallelogram, when AF the bafe of ‘ AG is greater than AB, and therefore AG exceeds AC the ‘ parallelogram defcribed upon AB in the fame angle, and be- ‘ tween the fame parallels, by the parallelogram BG.’

PROP. XXVII. THEOR.

OF all parallelograms applied to the fame ftraight line, and deficient by parallelograms, fimilar and fimi- larly fituated to that which is defcribed upon the half of the line; that which is applied to the half, and is fimilar to its defect, is the greateft.

Let AB be a ftraight line divided into two equal parts in C; and let the parallelogram AD be applied to the half AC, which is therefore deficient from the parallelogram upon the whole line AB by the parallelogram CE upon the other half CB: Of all the parallelograms applied to any other parts of AB,

AB, and deficient by parallelograms that are similar, and simi-
larly situated to CE, AD is the greatest.

Let AF be any parallelogram applied to AK, any other part
of AB than the half, so as to be deficient from the parallelo-
gram upon the whole line AB by the parallelogram KH simi-
lar, and similarly situated to CE; AD is greater than AF.

First, let AK, the base of AF, be greater than AC the half of
AB; and because CE is similar to the
parallelogram KH, they are about the
same diameter ᵃ: Draw their diame-
ter DB, and complete the scheme : Be-
cause the parallelogram CF is equal
ᵇ to FE, add KH to both, therefore
the whole CH is equal to the whole
KE :   But CH is equal ᶜ to CG, be-
cause the base AC is equal to the base
CB; therefore CG is equal to KE:
To each of these add CF; then the
whole AF is equal to the gnomon CHL : Therefore CE, or the
parallelogram AD, is greater than the parallelogram AF.

a 26. 6;

b 43. I.

c 36. I.
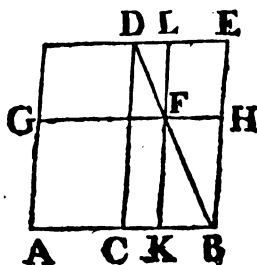
Next, let AK the base of AF,
be less than AC, and, the same
construction being made, the paral-
lelogram DH is equal to DG ᶜ, for
HM is equal to MG ᵈ, because BC
is equal to CA; wherefore DH is
greater than LG :   But DH is equal ᵇ
to DK ; therefore DK is greater than
LG : To each of these add AL ; then
the whole AD is greater than the
whole AF.   Therefore of all paralle-
lograms applied, &c.   Q. E. D.

d 34. I.
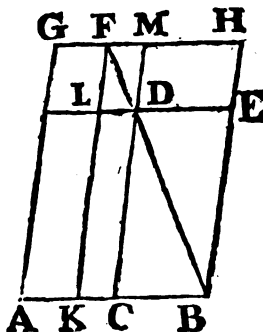
P R O P.

PROP. XXVIII. PROB.

See N.   TO a given ſtraight line to apply a parallelogram e-
qual to a given rectilineal figure, and deficient by
a parallelogram ſimilar to a given parallelogram : But the
given rectilineal figure to which the parallelogram to be
applied is to be equal, muſt not be greater than the pa-
rallelogram applied to half of the given line, having its
defect ſimilar to the defect of that which is to be ap-
plied; that is, to the given parallelogram.

Let AB be the given ſtraight line, and C the given rectili-
neal figure, to which the parallelogram to be applied is requi-
red to be equal, which figure muſt not be greater than the pa-
rallelogram applied to the half of the line having its defect from
that upon the whole line ſimilar to the defect of that which is
to be applied; and let D be the parallelogram to which this
defect is required to be ſimilar. It is required to apply a pa-
rallelogram to the ſtraight
line AB, which ſhall be equal
to the figure C, and be defi-
cient from the parallelogram
upon the whole line by a pa-
rallelogram ſimilar to D.

Divide AB into two equal
a 10. 1.   parts a in the point E, and
upon EB deſcribe the paral-
b 18. 6.   lelogram EBFG ſimilar b and
ſimilarly ſituated to D, and
complete the parallelogram
AG, which muſt either be e-
qual to C, or greater than it,
by the determination : And if
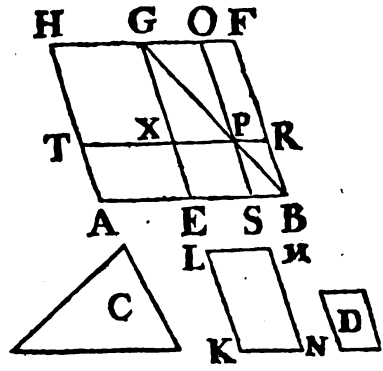
AG be equal to C, then what was required is already done :
For, upon the ſtraight line AB, the parallelogram AG is applied
equal to the figure C, and deficient by the parallelogram EF
ſimilar to D : But, if AG be not equal to C, it is greater than
it; and EF is equal to AG ; therefore EF alſo is greater than
c 25. 6.   C.   Make c the parallelogram KLMN equal to the exceſs of
EF above C, and ſimilar and ſimilarly ſituated to D; but D is
d. 21. 6.   ſimilar to EF, therefore d alſo KM is ſimilar to EF: Let KL
be

be the homologous fide to EG, and LM to GF : And becaufe EF is equal to C and KM together, EF is greater than KM ; therefore the ftraight line EG is greater than KL, and GF than LM: Make GX equal to LK, and GO equal to LM, and complete the parallelogram XGOP : Therefore XO is equal and fimilar to KM ; but KM is fimilar to EF ; wherefore alfo XO is fimilar to EF, and therefore XO and EF are about the fame diameter [e]: Let GPB be their diameter, and complete the fcheme : Then becaufe EF is equal to C and KM together, and XO a part of the one is equal to KM a part of the other, the remainder, viz. the gnomon ERO, is equal to the remainder C: And becaufe OR is equal [f] to XS, by adding SR to each, the whole OB is equal to the whole XB : But XB is equal [g] to TE, becaufe the bafe AE is equal to the bafe EB ; wherefore alfo TE is equal to OB : Add XS to each, then the whole TS is equal to the whole, viz. to the gnomon ERO: But it has been proved, that the gnomon ERO is equal to C, and therefore alfo TS is equal to C. Wherefore the parallelogram TS, equal to the given rectilineal figure C, is applied to the given ftraight line AB deficient by the parallelogram SR, fimilar to the given one D, becaufe SR is fimilar to EF [h]. Which was to be done.

e 26. 6.

f 34. 1.
g 36. 1.

h 24. 6.

PROP. XXIX. PROB.

TO a given ftraight line to apply a parallelogram equal to a given rectilineal figure, exceeding by a parallelogram fimilar to another given.

Let AB be the given ftraight line, and C the given rectilineal figure to which the parallelogram to be applied is required to be equal, and D the parallelogram to which the excefs of the one to be applied above that upon the given line is required to be fimilar. It is required to apply a parallelogram to the given ftraight line AB which fhall be equal to the figure C, exceeding by a parallelogram fimilar to D.

Divide AB into two equal parts in the point E, and upon EB defcribe [a] the parallelogram EL fimilar and fimilarly fitua- a 18. 6.
ted

b 25. 6.
c 21. 6.

ted to D : And make **b** the parallelogram GH equal to EL and C together, and similar and similarly situated to D ; wherefore GH is similar to EL **c** : Let KH be the side homologous to FL, and KG to FE : And because the parallelogram GH is greater than EL, therefore the side KH is greater than FL, and KG than FE : Produce FL and FE, and make FLM equal to KH, and FEN to KG, and complete the parallelogram MN.

MN is therefore equal and similar to GH ; but GH is similar to EL ; wherefore MN is similar to EL, and consequently EL and MN are about the same diameter **d** : Draw their diameter FX, and complete the scheme. Therefore, since GH is equal to EL and C together, and that GH is equal to MN ; MN is equal to EL and C :
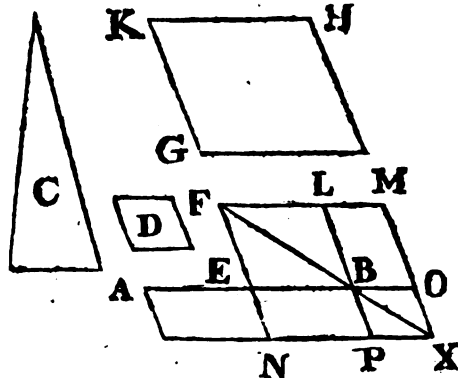
d 26. 6.

Take away the common part EL ; then the remainder, viz. the gnomon NOL, is equal to C. And because AE is equal to EB, the parallelogram AN is equal **e** to the parallelogram NB, that is, to BM **f**. Add NO to each ; therefore the whole, viz. the parallelogram AX, is equal to the gnomon NOL. But the gnomon NOL is equal to C ; therefore also AX is equal to C. Wherefore to the straight line AB there is applied the parallelogram AX equal to the given rectilineal C, exceeding by the parallelogram PO, which is similar to D, because PO is similar to EL **g**. Which was to be done.

e 36. 1.
f 43. 1.

g 24. 6.

PROP. XXX. PROB.

TO cut a given straight line in extreme and mean ratio.

Let AB be the given straight line ; it is required to cut it in extreme and mean ratio.

<div align="right">Upon</div>

Upon AB defcribe ᵃ the fquare BC, and to AC apply the
parallelogram CD equal to BC, exceeding by the figure AD fi-
milar to BC ᵇ : But BC is a fquare,
therefore alfo AD is a fquare; and be-
caufe BC is equal to CD, by taking the
common part CE from each, the re-
mainder BF is equal to the remainder
AD : And thefe figures are equiangu-
lar, therefore their fides about the equal
angles are reciprocally proportional c :
Wherefore, as FE to ED, fo AE to EB :
But FE is equal to AC ᵈ, that is, to AB;
and ED is equal to AE : Therefore as
BA to AE, fo is AE to EB : But AB is
greater than AE ; wherefore AE is
greater than EB ᵉ : Therefore the ftraight line AB is cut in ex-
treme and mean ratio in E ᶠ. Which was to be done.

a 46. 1.
b 29. 6.
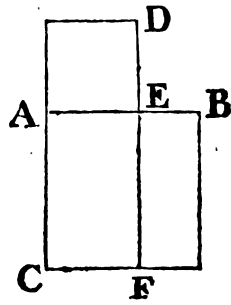
c 14. 6.

d 34. 1.

e 14. 5.
f 3. def. 6.

<div align="center">Otherwife,</div>

Let AB be the given ftraight line; it is required to cut it in
extreme and mean ratio.

Divide AB in the point C, fo that the rectangle contained
by AB, BC be equal to the fquare of AC g.
Then, becaufe the rectangle AB, BC is equal
to the fquare of AC, as BA to AC, fo is
AC to CB ʰ : Therefore AB is cut in extreme and mean ra-
tio in C ᶠ. Which was to be done.

g 11. 2.

h 17. 6.

PROP. XXXI. THEOR.

I N right angled triangles, the rectilineal figure defcri-
bed upon the fide oppofite to the right angle, is equal
to the fimilar, and fimilarly defcribed figures upon the
fides containing the right angle.

Let ABC be a right angled triangle, having the right angle
BAC : The rectilineal figure defcribed upon BC is equal to the
fimilar, and fimilarly defcribed figures upon BA, AC.
Draw the perpendicular AD ; therefore, becaufe in the
right angled triangle ABC, AD is drawn from the right angle
at A perpendicular to the bafe BC, the triangles ABD, ADC
are fimilar to the whole triangle ABC, and to one another ᵃ, a 8. 6.

<div align="right">and</div>

and becaufe the triangle ABC is fimilar to ADB, as CB to
BA, fo is BA to BD [b]; and becaufe thefe three ftraight lines
are proportionals, as the firft to the third, fo is the figure up-
on the firft to the fimilar, and fimilarly defcribed figure upon the

fecond [c] : Therefore as CB to
BD, fo is the figure upon
CB to the fimilar and fimi-
larly defcribed figure upon
BA : And, inverfely [d], as DB
to BC, fo is the figure upon
BA to that upon BC : For
the fame reafon, as DC to
CB, fo is the figure upon CA
to that upon CB.   Wherefore

as BD and DC together to BC, fo are the figures upon BA, AC
to that upon BC [e] : But BD and DC together are equal to BC.
Therefore the figure defcribed on BC is equal [f] to the fimilar
and fimilarly defcribed figures on BA, AC.   Wherefore, in
right angled triangles, &c. Q. E. D.
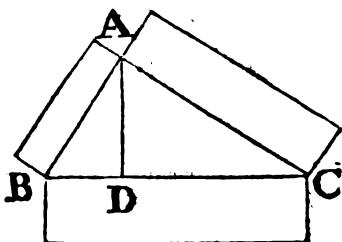
PROP. XXXII.   THEOR.

IF two triangles which have two fides of the one pro-
portional to two fides of the other, be joined at one
angle, fo as to have their homologous fides parallel to
one another ; the remaining fides fhall be in a ftraight
line.

Let ABC, DCE be two triangles which have the two fides
BA, AC proportional to the two CD, DE, viz. BA to AC,
as CD to DE ; and let AB be parallel to DC, and AC to DE.
BC and CE are in a ftraight line.

Becaufe AB is parallel to
DC, and the ftraight line
AC meets them, the al-
ternate angles BAC, ACD
are equal [a]; for the fame
reafon, the angle CDE is
equal to the angle ACD ;
wherefore alfo BAC is e-
qual to CDE : And becaufe

the
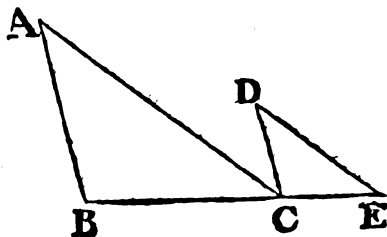
the triangles ABC, DCE have one angle at A equal to one at D, and the ſides about theſe angles proportionals, viz BA to AC, as CD to DE, the triangle ABC is equiangular [b] to DCE: And the angle ABC is equal to the angle DCE: And the angle BAC was proved to be equal to ACD: Therefore the whole angle ACE is equal to the two angles ABC, BAC; add the common angle ACB, then the angles ACE, ACB are equal to the angles ABC, BAC, ACB: But ABC, BAC, ACB are equal to two right angles [c]; therefore alſo the angles ACE, ACB are equal to two right angles: And ſince at the point C, in the ſtraight line AC, the two ſtraight lines BC, CE, which are on the oppoſite ſides of it, make the adjacent angles ACE, ACB equal to two right angles; therefore [d] BC and CE are in a ſtraight line. Wherefore, if two triangles, &c. Q. E. D.

b 6. 6.

c 32. 1.

d 14. 1.

PROP. XXXIII. THEOR.

IN equal circles, angles, whether at the centres or cir-cumferences, have the ſame ratio which the circum-ferences on which they ſtand have to one another: So alſo have the ſectors.

See N.

Let ABC, DEF be equal circles; and at their centres the angles BGC, EHF, and the angles BAC, EDF at their cir-cumferences; as the circumference BC to the circumference EF, ſo is the angle BGC to the angle EHF, and the angle BAC to the angle EDF; and alſo the ſector BGC to the ſector EHF.

Take any number of circumferences CK, KL, each equal to BC, and any number whatever FM, MN each equal to EF: And join GK, GL, HM, HN. Becauſe the circumferences BC, CK, KL are all equal, the angles BGC, CGK, KGL are alſo all equal [a]: Therefore what multiple ſoever the circum-ference BL is of the circumference BC, the ſame multiple is the angle BGL of the angle BGC: For the ſame reaſon, what-ever multiple the circumference EN is of the circumference EF, the ſame multiple is the angle EHN of the angle EHF:

And

a 27. 3.

And if the circumference BL be equal to the circumference
EN, the angle BGL is alfo equal [a] to the angle EHN; and
[a 27. 3.] if the circumference BL be greater than EN, likewife the angle
BGL is greater than EHN; and if lefs, lefs: There being then
four magnitudes, the two circumferences BC, EF, and the
two angles BGC, EHF; of the circumference BC, and of the
angle BGC, have been taken any equimultiples whatever, viz.
the circumference BL, and the angle BGL; and of the circum-
ference EF, and of the angle EHF, any equimultiples what-

ever, viz. the circumference EN, and the angle EHN: And
it has been proved, that, if the circumference BL be greater
than EN, the angle BGL is greater than EHN; and if e-
qual, equal; and if lefs, lefs: As therefore the circumference
[b 5. Def. 5.] BC to the circumference EF, fo [b] is the angle BGC to the
angle EHF: But as the angle BGC is to the angle EHF, fo is
[c 15. 6.] [c] the angle BAC to the angle EDF, for each is double of
[d 20. 3.] each [d]: Therefore, as the circumference BC is to EF, fo is the
angle BGC to the angle EHF, and the angle BAC to the angle
EDF.
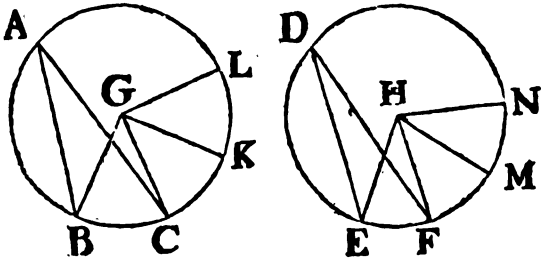
Alfo, as the circumference BC to EF, fo is the fector BGC
to the fector EHF.   Join BC, CK, and in the circumferences
BC, CK take any points X, O, and join BX, XC, CO, OK:
Then, becaufe in the triangles GBC, GCK the two fides BG,
GC are equal to the two CG, GK, and that they contain e-
[e 4. 1.] qual angles; the bafe BC is equal [e] to the bafe CK, and the
triangle GBC to the triangle GCK: And becaufe the circum-
ference BC is equal to the circumference CK, the remaining
part of the whole circumference of the circle ABC, is equal to
the remaining part of the whole circumference of the fame
circle: Wherefore the angle BXC is equal to the angle COK [a];
[f 11. def. 3.] and the fegment BXC is therefore fimilar to the fegment COK [f];
and

and they are upon equal ſtraight lines BC, BK : But ſimilar ſeg-
ments of circles upon equal ſtraight lines, are equal ᵍ to one ano-
ther : Therefore the ſegment B X C is equal to the ſegment COK : ᵍ 24. 3.
And the triangle BGC is equal to the triangle CGK ; therefore
the whole, the ſector BGC, is equal to the whole, the ſector CGK :
For the ſame reaſon, the ſector KGL is equal to each of the
ſectors BGC, CGK : In the ſame manner, the ſectors EHF,
FHM, MHN may be proved equal to one another : Therefore,
what multiple ſoever the circumference BL is of the circumfe-
rence BC, the ſame multiple is the ſector BGL of the ſector
BGC : For the ſame reaſon, whatever multiple the circumfe-
rence EN is of EF, the ſame multiple is the ſector EHN of the
ſector EHF : And if the circumference BL be equal to EN, the
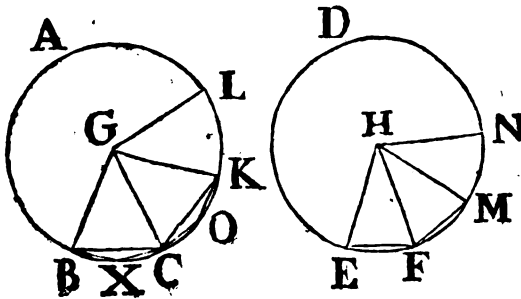
ſector BGL is equal to the ſector EHN ; and if the circumfe-
rence BL be greater than EN, the ſector BGL is greater than
the ſector EHN ; and if leſs, leſs : Since, then, there are four
magnitudes, the two circumferences BC, EF, and the two ſec-
tors BGC, EHF, and of the circumference BC, and ſector
BGC, the circumference BL and ſector BGL are any equi-
multiples whatever ; and of the circumference EF and ſector
EHF, the circumference EN and ſector EHN are any equi-
multiples whatever ; and that it has been proved, if the circum-
ference BL be greater than EN, the ſector BGL is greater than
the ſector EHN ; and if equal, equal ; and if leſs, leſs. There-
fore ᵇ, as the circumference BC is to the circumference EF, ſo ᵇ 5. def. 5.
is the ſector BGC to the ſector EHF. Wherefore, in equal cir-
cles, &c. Q. E. D.

N   P R O P.

PROP. B. THEOR.

IF an angle of a triangle be bisected by a straight line, which likewise cuts the base ; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line bisecting the angle.

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD ; the rectangle BA, AC is equal to the rectangle BD, DC, together with the square of AD.

a 5. 4.   Describe the circle a ACB about the triangle, and produce AD to the circumference in E,
and join EC : Then, because the angle BAD is equal to the angle CAE, and the angle ABD to the
b 21. 3. angle b AEC, for they are in the same segment; the triangles ABD, AEC are equiangular to one another : Therefore as BA to AD,
c 4. 6. so is c EA to AC, and consequently the rectangle BA, AC is
d 16. 6. equal d to the rectangle EA, AD,
e 3. 2. that is e, to the rectangle ED, DA together with the square of AD : But the rectangle ED, DA
f 35. 3. is equal to the rectangle f BD, DC. Therefore the rectangle BA, AC is equal to the rectangle BD, DC, together with the square of AD. Wherefore, if an angle, &c. Q. E. D.
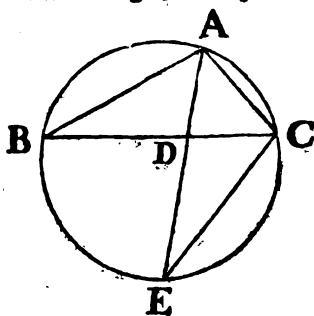
PROP. C. THEOR.

IF from any angle of a triangle a straight line be drawn perpendicular to the base ; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

Let ABC be a triangle, and AD the perpendicular from the angle A to the base BC ; the rectangle BA, AC is equal to the rectangle contained by AD and the diameter of the circle described about the triangle.

Describe

Defcribe ^a the circle ACB about the triangle, and draw its diameter AE, and join EC: Becaufe the right angle BDA is equal ^b to the angle ECA in a femicircle, and the angle ABD to the angle AEC in the fame fegment ^c; the triangles ABD, AEC are equiangular: Therefore as ^d BA to AD, fo is EA to AC; and confequently the rectangle BA, AC is equal ^e to the rectangle EA, AD. If therefore, from an angle, &c. Q. E. D.
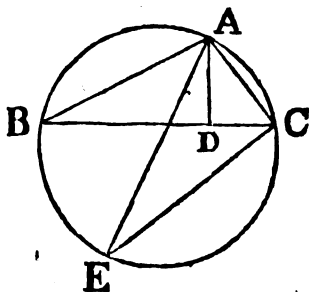
P R O P. D.   T H E O R.

THE rectangle contained by the diagonals of a quadrilateral infcribed in a circle, is equal to both the rectangles contained by its oppofite fides.

Let ABCD be any quadrilateral infcribed in a circle, and join AC, BD; the rectangle contained by AC, BD is equal to the two rectangles contained by AB, CD, and by AD, BC †.

Make the angle ABE equal to the angle DBC; add to each of thefe the common angle EBD, then the angle ABD is equal to the angle EBC: And the angle BDA is equal ^a to the angle BCE, becaufe they are in the fame fegment; therefore the triangle ABD is equiangular to the triangle BCE: Wherefore ^b as BC is to CE, fo is BD to DA; and confequently the rectangle BC, AD is equal ^c to the rectangle BD, CE: Again, becaufe the angle ABE is equal to the angle DBC, and the angle BAE to the angle BDC, the triangle ABE is equiangular to the triangle BCD: As therefore BA to AE, fo is BD to DC; wherefore the rectangle BA, DC is equal to the rectangle BD, AE: But the rectangle BC, AD has been fhewn equal to the rectangle BD, CE; therefore the whole rectangle AC, BD ^d is equal to the rectangle AB, DC, together with the rectangle AD, BC. Therefore the rectangle, &c. Q. E. D.
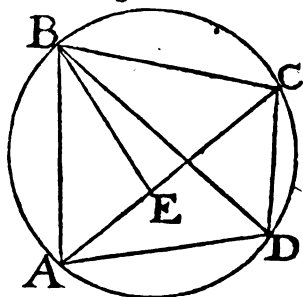
N 2   THE

† This is a Lemma of Cl. Ptolomæus, in page 9. of his μεγαλη συνταξις.

THE

ELEMENTS

OF

EUCLID,

BOOK XI.

DEFINITIONS.

I.

A Solid is that which hath length, breadth, and thickness.

II.

That which bounds a folid is a fuperficies.

III.

A ftraight line is perpendicular, or at right angles to a plane, when it makes right angles with every ftraight line meeting it in that plane.

IV.

A plane is perpendicular to a plane, when the ftraight lines drawn in one of the planes perpendicularly to the common fection of the two planes, are perpendicular to the other plane.

V.

The inclination of a ftraight line to a plane is the acute angle contained by that ftraight line, and another drawn from the point in which the firft line meets the plane, to the point in which a perpendicular to the plane drawn from any point of the firft line above the plane, meets the fame plane.

VI.

The inclination of a plane to a plane is the acute angle contained by two ftraight lines drawn from any the fame point of their common fection at right angles to it, one upon one plane, and the other upon the other plane.

VII. Two

VII.

Two planes are said to have the same, or a like inclination to one another, which two other planes have, when the said angles of inclination are equal to one another.

VIII.

Parallel planes are such which do not meet one another though produced.

IX.

A solid angle is that which is made by the meeting of more See N. than two plane angles, which are not in the same plane, in one point.

X.

' The tenth definition is omitted for reasons given in the notes.' See N.

XI.

Similar solid figures are such as have all their solid angles equal, See N. each to each, and which are contained by the same number of similar planes.

XII.

A pyramid is a solid figure contained by planes that are constituted betwixt one plane and one point above it in which they meet.

XIII.

A prism is a solid figure contained by plane figures of which two that are opposite are equal, similar, and parallel to one another ; and the others parallelograms.

XIV.

A sphere is a solid figure described by the revolution of a semicircle about its diameter, which remains unmoved.

XV.

The axis of a sphere is the fixed straight line about which the semicircle revolves.

XVI.

The centre of a sphere is the same with that of the semicircle.

XVII.

The diameter of a sphere is any straight line which passes thro' the centre, and is terminated both ways by the superficies of the sphere.

XVIII.

A cone is a solid figure described by the revolution of a right angled triangle about one of the sides containing the right angle, which side remains fixed.

If the fixed side be equal to the other side containing the right angle, the cone is called a right angled cone ; if it be less than the other side, an obtuse angled, and if greater, an acute angled cone.

XIX. The

XIX.

The axis of a cone is the fixed ſtraight line about which the triangle revolves.

XX.

The baſe of a cone is the circle deſcribed by that ſide containing the right angle, which revolves.

XXI.

A cylinder is a ſolid figure deſcribed by the revolution of a right angled parallelogram about one of its ſides which remains fixed.

XXII.

The axis of a cylinder is the fixed ſtraight line about which the parallelogram revolves.

XXIII.

The baſes of a cylinder are the circles deſcribed by the two revolving oppoſite ſides of the parallelogram.

XXIV.

Similar cones and cylinders are thoſe which have their axes and the diameters of their baſes proportionals.

XXV.

A cube is a ſolid figure contained by ſix equal ſquares.

XXVI.

A tetrahedron is a ſolid figure contained by four equal and equilateral triangles.

XXVII.

An octahedron is a ſolid figure contained by eight equal and equilateral triangles.

XXVIII.

A dodecachedron is a ſolid figure contained by twelve equal pentagons which are equilateral and equiangular.

XXIX.

An icoſahedron is a ſolid figure contained by twenty equal and equilateral triangles.

D E F.   A.

A parallelopiped is a ſolid figure contained by ſix quadrilateral figures whereof every oppoſite two are parallel.

P R O P.

PROP. I. THEOR.

ONE part of a ſtraight line cannot be in a plane and See N.
another part above it.

If it be poſſible, let AB, part of the ſtraight line ABC, be in
the plane, and the part BC above it :   And ſince the ſtraight
line AB is in the plane, it can be
produced in that plane : Let it be
produced to D : And let any plane
paſs thro' the ſtraight line AD, and
be turned about it until it paſs
thro' the point C ; and becauſe the
points B, C, are in this plane, the ſtraight line BC is in it [a] : a 7. def. 1.
Therefore there are two ſtraight lines ABC, ABD in the ſame
plane that have a common ſegment AB, which is impoſſible [b]. b. Cor.11.1.
Therefore one part, &c.   Q. E. D.
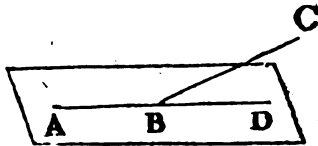
PROP. II.   THEOR.

TWO ſtraight lines which cut one another are in one See N.
plane, and three ſtraight lines which meet one ano-
ther are in one plane.

Let two ſtraight lines AB, CD cut one another in E ; AB,
CD are one plane :   And three ſtraight lines EC, CB, BE
which meet one another, are in one plane.
Let any plane paſs through the ſtraight
line EB, and let the plane be turned a-
bout EB, produced, if neceſſary, until it
paſs through the point C : Then becauſe
the points E, C are in this plane, the
ſtraight line EC is in it [a] : For the ſame   a 7. def. 1.
reaſon, the ſtraight line BC is in the
ſame ; and, by the hypotheſis, EB is in
it : Therefore the three ſtraight lines EC,
CB, BE are in one plane : But in the plane
in which EC, EB are, in the ſame are [b]   b. 1. 11.
CD, AB :   Therefore AB, CD are in
one plane.   Wherefore two ſtraight lines, &c.   Q. E. D.
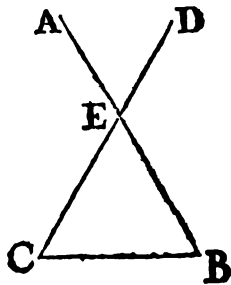
P R O P. III.   T H E O R.

See N. IF two planes cut one another, their common fection is
a ftraight line.

Let two planes AB, BC, cut one another, and let the line
DB be their common fection : DB is a
ftraight line : If it be not, from the point
D to B, draw, in the plane AB, the
ftraight line DEB, and in the plane BC
the ftraight line DFB : Then two ftraight
lines DEB, DFB have the fame extremi-
ties, and therefore include a fpace be-
a 10. Ax. 1. twixt them ; which is impoffible[a] : There-
fore BD the common fection of the planes
AB, BC cannot but be a ftraight line.
Wherefore, if two planes, &c. Q. E. D.
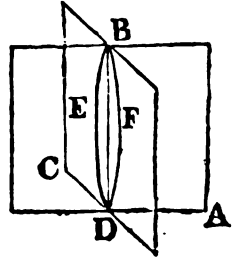
P R O P. IV.   T H E O R.

See N. IF a ftraight line ftand at right angles to each of two
ftraight lines in the point of their interfection, it fhall
alfo be at right angles to the plane which paffes through
them, that is, to the plane in which they are.

Let the ftraight line EF ftand at right angles to each of the
ftraight lines AB, CD in E, the point of their interfection : EF
is alfo at right angles to the plane paffing through AB, CD.
Take the ftraight lines AE, EB, CE, ED all equal to one an-
other ; and through E draw, in the plane in which are AB, CD,
any ftraight line GFH ; and join AD, CB ; then, from any
point F in EF, draw FA, FG, FD, FC, FH, FB : And becaufe
the two ftraight lines, AE, ED are equal to the two BE, EC,
a 15. 1.   and that they contain equal angles[a] AED, BEC, the bafe AD
b 4. 1.   is equal[b] to the bafe BC, and the angle DAE to the angle
EBC : And the angle AEG is equal to the angle BEH[a] ; there-
fore the triangles AEG, BEH have two angles of one equal
to two angles of the other, each to each, and the fides AE,
EB, adjacent to the equal angles, equal to one another ; where-
c 26. 1.   fore they fhall have their other fides equal[c]: GE is therefore
equal

equal to EH, and AG to BH : And becaufe AE is equal to EB, Book XI.
and FE common and at right angles to them, the bafe AF is
equal b to the bafe FB ; for the fame reafon, CF is equal to b 4. 1.
FD : And becaufe AD is equal to BC, and AF to FB, the two
fides FA, AD are equal to the two
FB, BC, each to each ; and the bafe DF
was proved equal to the bafe FC ;
therefore the angle FAD is equal d to   d 8. 1.
the angle FBC : Again, it was proved
that GA is equal to BH, and alfo AF
to FB ; FA, then, and AG, are equal
to FB and BH, and the angle FAG
has been proved equal to the angle
FBH ; therefore the bafe GF is equal
b to the bafe FH : Again, becaufe it
was proved, that GE is equal to EH,
and EF is common ; GE, EF are e-
qual to HE, EF; and the bafe GF
is equal to the bafe FH ; therefore the angle GEF is equal d to
the angle HEF ; and confequently each of thefe angles is a
right e angle. Therefore FE makes right angles with GH, e 10. def. 1.
that is, with any ftraight line drawn through E in the plane
paffing through AB, CD. In like manner, it may be proved, that
FE makes right angles with every ftraight line which meets it
in that plane. But a ftraight line is at right angles to a plane
when it makes right angles with every ftraight line which meets
it in that plane f :   Therefore EF is at right angles to the plane f 3. def. 11.
in which are AB, CD.   Wherefore, if a ftraight line, &c.
Q. E. D.
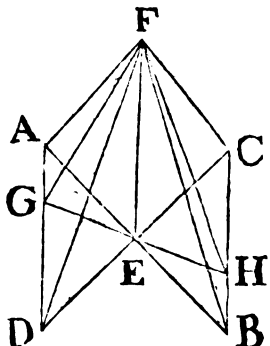
P R O P. V. T H E O R.

IF three ftraight lines meet all in one point, and a See N.
ftraight line ftands at right angles to each of them in
that point ; thefe three ftraight lines are in one and the
fame plane.

Let the ftraight line AB ftand at right angles to each of the
ftraight lines BC, BD, BE, in B the point where they meet ;
BC, BD, BE are in one and the fame plane.
It not, let, if it be poffible, BD and BE be in one plane, and
BC be above it ; and let a plane pafs through AB, BC, the
common fection of which with the plane, in which BD and BE
are,

are, fhall be a ftraight ᵃ line ; let this be BF : Therefore the three ftraight lines AB, BC, BF are all in one plane, viz. that which paffes through AB, BC ; and becaufe AB ftands at right angles to each of the ftraight lines BD, BE, it is alfo at right angles ᵇ to the plane paffing through them ; and therefore makes right angles ᶜ with every ftraight line meeting in that plane ; but BF which is in that plane meets it : Therefore the angle ABF is a right angle ; but the angle ABC, by the hypothefis, is alfo a right angle ; therefore the angle ABF is equal to the angle ABC, and they are both in the fame plane, which is impoffible : Therefore the ftraight line BC is not above the plane in which are BD and BE : Wherefore the three ftraight lines BC, BD, BE are in one and the fame plane. Therefore, if three ftraight lines, &c. Q. E. D.
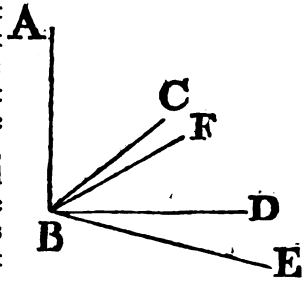
PROP. VI. THEOR.

IF two ftraight lines be at right angles to the fame plane, they fhall be parallel to one another.

Let the ftraight lines AB, CD be at right angles to the fame plane ; AB is parallel to CD.

Let them meet the plane in the points B, D, and draw the ftraight line BD, to which draw DE at right angles, in the fame plane ; and make DE equal to AB, and join BE, AE, AD. Then, becaufe AB is perpendicular to the plane, it ᵃ def. 11. shall make right ᵃ angles with every ftraight line which meets it, and is in that plane : But BD, BE, which are in that plane, do each of them meet AB . Therefore each of the angles ABD, ABE is a right angle : For the fame reafon, each of the angles CDB, CDE is a right angle : And becaufe AB is equal to DE, and BD common, the two fides AB, BD, are equal to the two ED, DB ; and they contain right angles ; therefore the bafe ᵇ 4. 1. AD is equal ᵇ to the bafe BE : Again, becaufe AB is equal
to

Book XI.
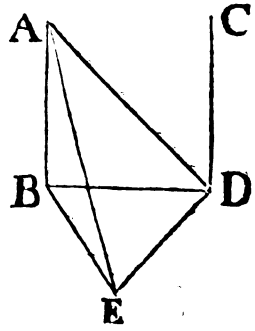
to DE, and BE to AD; AB, BE are equal to ED, DA; and, in the triangles ABE, EDA, the bafe AE is common; therefore the angle ABE is equal[c] to the angle EDA: But ABE is a right angle; therefore EDA is alfo a right angle, and ED perpendicular to DA: But it is alfo perpendicular to each of the two BD, DC: Wherefore ED is at right angles to each of the three ftraight lines BD, DA, DC in the point in which they meet: Therefore thefe three ftraight lines are all in the fame plane[d]: But AB is in the plane in which are BD, DA, becaufe any three ftraight lines which meet one another are in one plane[e]: Therefore AB, BD, DC are in one plane: And each of the angles ABD, BDC is a right angle; therefore AB is parallel[f] to CD. Wherefore, if two ftraight lines, &c. Q. E. D.

c 8. 1.

d 5. 11.

e 2. 11.

f 28. 1.

PROP. VII. THEOR.

IF two ftraight lines be parallel, the ftraight line drawn from any point in the one to any point in the other, is in the fame plane with the parallels.

See N.

Let AB, CD be parallel ftraight lines, and take any point E in the one, and the point F in the other: The ftraight line which joins E and F is in the fame plane with the parallels.

If not, let it be, if poffible, above the plane, as EGF; and in the plane ABCD in which the parallels are, draw the ftraight line EHF from E to F; and fince EGF alfo is a ftraight line, the two ftraight lines EHF, EGF include a fpace between them, which is impoffible[a]. Therefore the ftraight line joining the points E, F is not above the plane in which the parallels AB, CD are, and is therefore in that plane. Wherefore, if two ftraight lines, &c. Q. E. D.
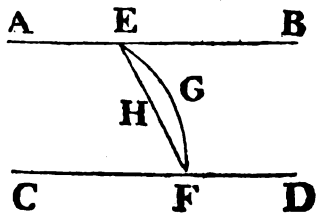
a 10. Ax. 1.

PROP. VIII. THEOR.

IF two ftraight lines be parallel, and one of them is at right angles to a plane; the other alfo fhall be at right angles to the fame plane.

See N.

Let

Let AB, CD be two parallel ſtraight lines, and let one of them AB be at right angles to a plane; the other CD is at right angles to the ſame plane.

Let AB, CD meet the plane in the points B, D, and join
g 7. 11. BD: Therefore ᵍ AB, CD, BD are in one plane. In the plane to which AB is at right angles, draw DE at right angles to BD, and make DE equal to AB, and join BE, AE, AD. And becauſe AB is perpendicular to the plane, it is perpendicular to every ſtraight line which meets it, and is in that
a 3. def 11. plane ᵃ: Therefore each of the angles ABD, ABE, is a right angle: And becauſe the ſtraight line BD meets the parallel ſtraight lines AB, CD; the angles ABD, CDB are together
b 29. 1. equal ᵇ to two right angles: And ABD is a right angle; therefore alſo CDB is a right angle, and CD perpendicular to BD: And becauſe AB is equal to DE, and BD common, the two AB, BD, are equal to the two ED, DB, and the angle ABD is equal to the angle EDB, becauſe each of them is a right angle; therefore the baſe AD
c 4. 1. is equal ᶜ to the baſe BE: Again, becauſe AB is equal to DE, and BE to AD; the two AB, BE are equal to the two ED, DA; and the baſe AE is common to the triangles ABE, EDA;
d 8. 1. wherefore the angle ABE is equal ᵈ to the angle EDA: And ABE is a right angle; and therefore EDA is a right angle, and ED perpendicular to DA:
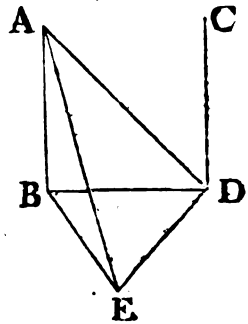
But it is alſo perpendicular to BD; therefore ED is perpendi-
e 4. 11. cular ᵉ to the plane which paſſes through BD, DA, and ſhall ᶠ
f 3. def. 11. make right angles with every ſtraight line meeting it in that plane: But DC is in the plane paſſing through BD, DA, becauſe all three are in the plane in which are the parallels AB, CD: Wherefore ED is at right angles to DC; and therefore CD is at right angles to DE: But CD is alſo at right angles to DB; CD then is at right angles to the two ſtraight lines DE, DB in the point of their interſection D; and therefore is at right angles ᵉ to the plane paſſing through DE, DB, which is the ſame plane to which AB is at right angles. Therefore, if two ſtraight lines, &c. Q. E. D.

PROP.

P R O P. IX. T H E O R.

TWO ftraight lines which are each of them parallel
to the fame ftraight line, and not in the fame plane
with it, are parallel to one another.

Let AB, CD be each of them parallel to EF, and not in the
fame plane with it; AB fhall be parallel to CD.
In EF take any point G, from which draw, in the plane
paffing through EF, AB, the ftraight line GH at right angles to
EF; and in the plane paffing through EF, CD, draw GK at
right angles to the fame EF. And
becaufe EF is perpendicular both
to GH and GK, EF is perpendi-
cular [a] to the plane HGK paffing
through them: And EF is parallel
to AB; therefore AB is at right
angles [b] to the plane HGK. For the
fame reafon, CD is likewife at right
angles to the plane HGK. There-
fore AB, CD are each of them at right angles to the plane
HGK. But if two ftraight lines be at right angles to the fame
plane, they fhall be parallel [c] to one another. Therefore AB is
parallel to CD. Wherefore two ftraight lines, &c. Q. E. D.

a 4. 11.

b 8. 11.

c 6. 11.
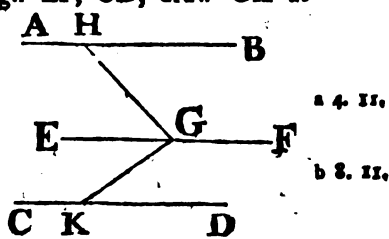
P R O P. X. T H E O R.

IF two ftraight lines meeting one another be parallel to
two others that meet one another, and are not in the
fame plane with the firft two; the firft two and the other
two fhall contain equal angles.

Let the two ftraight lines AB, BC which meet one another
be parallel to the two ftraight lines DE, EF that meet one an-
other, and are not in the fame plane with AB, BC. The angle
ABC is equal to the angle DEF.
Take BA, BC, ED, EF all equal to one another; and join
AD,

a 33. I.

b 9. 11.

c 1. Ax. 1.

d 8. 1.

AD, CF, BE, AC, DF: Becaufe BA is equal and parallel to ED, therefore AD is [a] both equal and parallel to BE. For the fame reafon, CF is equal and parallel to BE. Therefore AD and CF are each of them equal and parallel to BE. But ftraight lines that are parallel to the fame ftraight line, and not in the fame plane with it, are parallel [b] to one another. Therefore AD is parallel to CF; and it is equal [c] to it, and AC, DF join them towards the fame parts; and therefore [a] AC is equal and parallel to DF. And becaufe AB, BC are equal to DE, EF, and the bafe AC to the bafe DF; the angle ABC is equal [d] to the angle DEF. Therefore, if two ftraight lines, &c. Q. E. D.
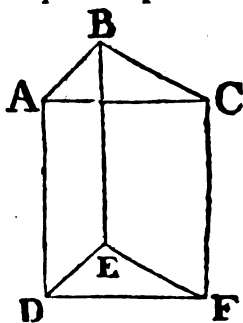
PROP. XI. PROB.

TO draw a ftraight line perpendicular to a plane, from a given point above it.

Let A be the given point above the plane BH; it is required to draw from the point A a ftraight line perpendicular to the plane BH.

a 12. 1.

b 11. 1.

c 31. 1.

d 4. 11.

e 8. 11.

f 3. def. 11.

In the plane draw any ftraight line BC, and from the point A draw [a] AD perpendicular to BC. If then AD be alfo perpendicular to the plane BH, the thing required is already done; but if it be not, from the point D draw [b], in the plane BH, the ftraight line DE at right angles to BC; and from the point A draw AF perpendicular to DE; and through F draw [c] GH parallel to BC: And becaufe BC is at right angles to ED and DA, BC is at right angles [d] to the plane paffing through ED, DA. And GH is parallel to BC; but, if two ftraight lines be parallel, one of which is at right angles to a plane, the other fhall be at right [e] angles to the fame plane; wherefore GH is at right angles to the plane through ED, DA, and is perpendicular [f] to every ftraight line meeting it in that plane. But AF, which is in the plane through ED, DA, meets it :
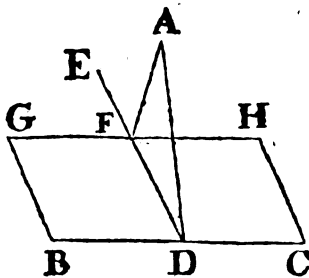
it : Therefore GH is perpendicular to AF; and confequently AF is perpendicular to GH; and AF is perpendicular to DE; therefore AF is perpendicular to each of the ftraight lines GH, DE. But if a ftraight line ftands at right angles to each of two ftraight lines in the point of their interfection, it fhall alfo be at right angles to the plane paffing through them. But the plane paffing through ED, GH is the plane BH; therefore AF is perpendicular to the plane BH. Therefore, from the given point A, above the plane BH, the ftraight line AF is drawn perpendicular to that plane. Which was to be done.

PROP. XII. PROB.

TO erect a ftraight line at right angles to a given plane, from a point given in the plane.

Let A be the point given in the plane; it is required to erect a ftraight line from the point A at right angles to the plane.

From any point B above the plane draw [a] BC perpendicular to it; and from A draw [b] AD parallel to BC. Becaufe, therefore, AD, CB are two parallel ftraight lines, and one of them BC is at right angles to the given plane, the other AD is alfo at right angles to

a 11. 11.
b 31. 1.

it [c]. Therefore a ftraight line has been erected at right angles to a given plane from a point given in it. Which was to be done.
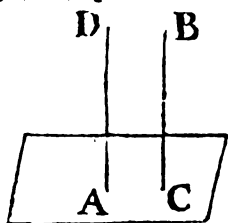
c 8. 11.

PROP. XIII. THEOR.

FROM the fame point in a given plane, there cannot be two ftraight lines at right angles to the plane, upon the fame fide of it: And there can be but one perpendicular to a plane from a point above the plane.

For, if it be poffible, let the two ftraight lines AC, AB, be at right angles to a given plane from the fame point A in the plane, and upon the fame fide of it; and let a plane pafs through BA, AC; the common fection of this with the given plane is a ftraight line

a line paſſing through A: Let DAE be their common ſection:
Therefore the ſtraight lines AB, AC, DAE are in one plane:
And becauſe CA is at right angles to the given plane, it ſhall
make right angles with every ſtraight
line meeting it in that plane. But
DAE, which is in that plane, meets
CA; therefore CAE is a right angle.
For the ſame reaſon BAE is a right
angle. Wherefore the angle CAE
is equal to the angle BAE; and
they are in one plane, which is
impoſſible. Alſo, from a point a-
bove a plane, there can be but one perpendicular to that plane;

† 6. 11. for, if there could be two, they would be parallel ᵇ to one
another, which is abſurd. Therefore, from the ſame point, &c.
Q. E. D.
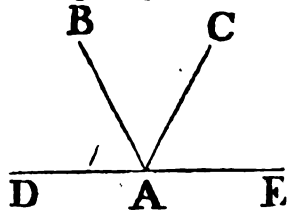
PROP. XIV. THEOR.

PLANES to which the ſame ſtraight line is perpendi-
cular, are parallel to one another.

Let the ſtraight line AB be perpendicular to each of the
planes CD, EF; theſe planes are parallel to one another.
If not, they ſhall meet one another when produced; let
them meet; their common ſection
ſhall be a ſtraight line GH, in which
take any point K, and join AK, BK:
Then, becauſe AB is perpendicular to

a 3. def. 11. the plane EF, it is perpendicular ᵃ to
the ſtraight line BK which is in that
plane. Therefore ABK is a right angle.
For the ſame reaſon, BAK is a right
angle; wherefore the two angles ABK,
BAK of the triangle ABK are equal
to two right angles, which is impoſ-

b 17. 1. ſible ᵇ: Therefore the planes CD, EF,
though produced, do not meet one an-

c 8. def. 11. other ; that is, they are parallel ᶜ.
Therefore planes, &c. Q. E. D.
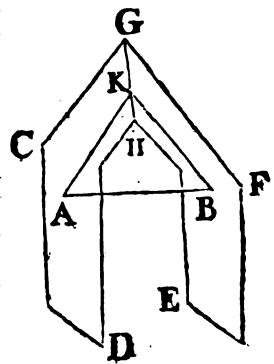
PROP.

P R O P. XV. T H E O R.

IF two ſtraight lines meeting one another, be parallel to two ſtraight lines which meet one another, but are not in the ſame plane with the firſt two; the plane which paſſes through theſe is parallel to the plane paſſing through the others.

Let AB, BC, two ſtraight lines meeting one another, be parallel to DE, EF that meet one another, but are not in the ſame plane with AB, BC : The planes through AB, BC, and DE, EF ſhall not meet, though produced.

From the point B draw BG perpendicular ^a to the plane ^a 11. 11. which paſſes through DE, EF, and let it meet that plane in G; and through G draw GH parallel ^b to ED, and GK pa- ^b 31. 1. rallel to EF : And becauſe BG is perpendicular to the plane through DE, EF, it ſhall make right angles with every ſtraight line meeting it in that plane ^c : But the ſtraight lines ^c 3. def. 11. GH, GK in that plane meet it : Therefore each of the an- gles BGH, BGK is a right angle : And becauſe BA is parallel ^d to GH (for each of ^d 9. 11. them is parallel to DE, and they are not both in the ſame plane with it) the angles GBA, BGH are together equal ^e to two right angles : And BGH is a ^e 29. 1. right angle ; therefore alſo GBA is a right angle, and GB per- pendicular to BA : For the ſame reaſon, GB is perpendicular to BC : Since therefore the ſtraight line GB ſtands at right an- gles to the two ſtraight lines BA, BC, that cut one another in B ; GB is perpendicular ^f to the plane through BA, BC : And ^f 4. 11. it is perpendicular to the plane through DE, EF ; therefore BG is perpendicular to each of the planes through AB, BC, and DE, EF : But planes to which the ſame ſtraight line is perpendicular, are parallel ^g to one another : Therefore the plane through AB, ^g 14. 11. BC is parallel to the plane through DE, EF. Wherefore, if two ſtraight lines, &c. Q. E. D.

O   P R O P.
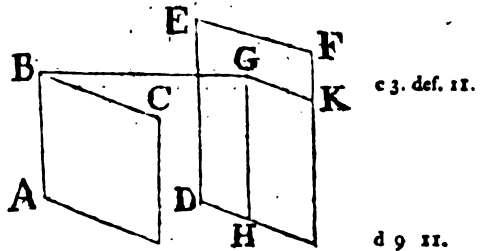
PROP. XVI. THEOR.

IF two parallel planes be cut by another plane, their common fections with it are parallels.

Let the parallel planes, AB, CD be cut by the plane EFHG, and let their common fections with it be EF, GH : EF is parallel to GH.

For, if it is not, EF, GH fhall meet, if produced, either on the fide of FH, or EG : Firft, let them be produced on the fide of FH, and meet in the point K : Therefore, fince EFK is in the plane AB, every point in EFK is in that plane; and K is a point in EFK ; therefore K is in the plane AB: For the fame reafon K is alfo in the plane CD : Wherefore the planes AB, CD produced meet one another ; but they do not meet, fince they are parallel by the hypothefis : Therefore the ftraight lines EF, GH do not meet when produced on the fide of FH : In the fame manner it may be proved, that EF, GH do not meet when produced on the fide of EG : But ftraight lines which are in the fame plane and do not meet, though produced either way, are parallel : Therefore EF is parallel to GH. Whcrefore, if two parallel planes, &c. Q. E. D.
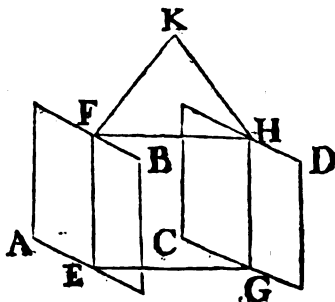
PROP. XVII. THEOR.

IF two ftraight lines be cut by parallel planes, they fhall be cut in the fame ratio.

Let the ftraight lines AB, CD be cut by the parallel planes GH, KL, MN, in the points A, E, B ; C, F, D : As AE is to EB, fo is CF to FD.

Join AC, BD, AD, and let AD meet the plane KL in the point X ; and join EX, XF : Becaufe the two parallel planes KL, MN are cut by the plane EBDX, the common fections

EX,

EX, BD, are parallel [a]. For the fame reafon, becaufe the two parallel planes GH, KL are cut by the plane AXFC, the common fections AC, XF are parallel: And becaufe EX is parallel to BD, a fide of the triangle ABD, as AE to EB, fo is [b] AX to XD. Again, becaufe XF is parallel to AC, a fide of the triangle ADC, as AX to XD, fo is CF to FD: And it was proved that AX is to XD, as AE to EB; Therefore c; as AE to EB, fo. is CF to FD. Wherefore, if two ftraight lines, &c. Q. E. D.
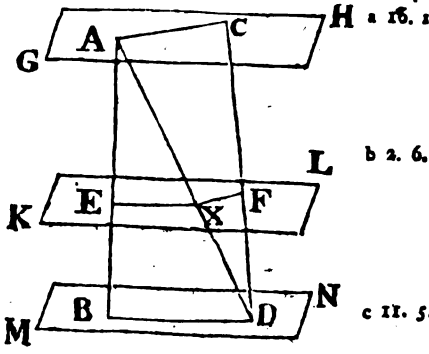
P R O P. XVIII. T H E O R.

IF a ftraight line be at right angles to a plane, every plane which paffes through it fhall be at right angles to that plane.

Let the ftraight line AB be at right angles to a plane CK; every plane which paffes through AB fhall be at right angles to the plane CK.

Let any plane DE pafs through AB, and let CE be the common fection of the planes DE, CK; take any point F in CE, from which draw FG in the plane DE at right angles to CE: And becaufe AB is perpendicular to the plane CK, therefore it is alfo perpendicular to every ftraight line in that plane meeting it [a]; And confequently it is perpendicular to CE: Wherefore ABF is a right angle; but GFB is likewife a right angle; therefore AB is parallel [b] to FG. And AB is at right angles to the plane CK; therefore FG is alfo at right angles to the fame plane c. But one plane is at right angles to another plane when the ftraight lines drawn in one of the planes, at right angles

a 3. def. 11.

b 28. 1.

c 8. 11.

to
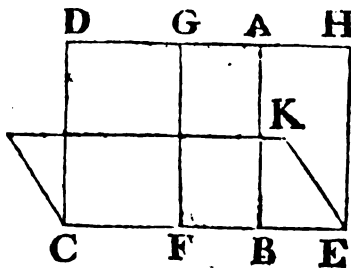
to their common section, are also at right angles to the other plane [d]; and any straight line FG in the plane DE, which is at right angles to CE the common section of the planes, has been proved to be perpendicular to the other plane CK; therefore the plane DE is at right angles to the plane CK. In like manner, it may be proved that all the planes which pass through AB are at right angles to the plane CK. Therefore, if a straight line, &c. Q. E. D.

PROP. XIX. THEOR.

IF two planes cutting one another be each of them perpendicular to a third plane ; their common section shall be perpendicular to the same plane.

Let the two planes AB, BC be each of them perpendicular to a third plane, and let BD be the common section of the first two; BD is perpendicular to the third plane.

If it be not, from the point D draw, in the plane AB, the straight line DE at right angles to AD the common section of the plane AB with the third plane; and in the plane BC draw DF at right angles to CD the common section of the plane BC with the third plane. And because the plane AB is perpendicular to the third plane, and DE is drawn in the plane AB at right angles to AD their common section, DE is perpendicular to the third

plane [a]. In the same manner, it may be proved that DF is perpendicular to the third plane. Wherefore, from the point D two straight lines stand at right angles to the third plane, upon the same

side of it, which is impossible [b] : Therefore, from the point D there cannot be any straight line at right angles to the third plane, except BD the common section of the planes AB, BC. BD therefore is perpendicular to the third plane. Wherefore, if two planes, &c. Q. E. D.
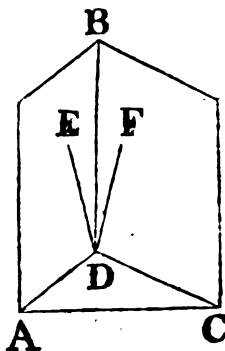
PROP.

P R O P. XX. T H E O R.

IF a folid angle be contained by three plane angles, any See N.
two of them are greater than the third.

Let the folid angle at A be contained by the three plane an-
gles BAC, CAD, DAB. Any two of them are greater than
the third.

If the angles BAC, CAD, DAB be all equal, it is evident
that any two of them are greater than the third. But if they
are not, let BAC be that angle which is not lefs than either of
the other two, and is greater than one of them DAB; and
at the point A in the ftraight line AB, make, in the plane
which paffes through BA, AC, the angle BAE equal ᵃ to the a 23. 1.
angle DAB; and make AE equal to AD, and through E
draw BEC cutting AB, AC in the
points B, C, and join DB, DC. And
becaufe DA is equal to AE, and AB
is common, the two DA, AB are e-
qual to the two EA, AB, and the
angle DAB is equal to the angle EAB:
Therefore the bafe DB is equal ᵇ to
the bafe BE. And becaufe BD, DC
are greater ᶜ than CB, and one of them
BD has been proved equal to BE a part of CB, therefore the
other DC is greater than the remaining part EC.   And becaufe
DA is equal to AE, and AC common, but the bafe DC greater
than the bafe EC; therefore the angle DAC is greater ᵈ than d. 25. 1.
the angle EAC; and, by the conftruction, the angle DAB is
equal to the angle BAE; wherefore the angles DAB, DAC are
together greater than BAE, EAC, that is, than the angle BAC.
But BAC is not lefs than either of the angles DAB, DAC;
therefore BAC, with either of them, is greater than the other.
Wherefore, if a folid angle, &c. Q. E. D.
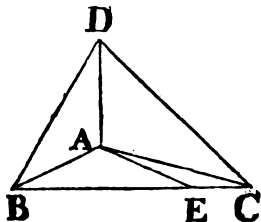
b 4. 1.

c 20. 1.

P R O P. XXI. T H E O R.

EVERY folid angle is contained by plain angles which
together are lefs than four right angles.

Firft, Let the folid angle at A be contained by three plane
angles BAC, CAD, DAB. Thefe three together are lefs than
four right angles.

O 3   Take

Take in each of the ſtraight lines AB, AC, AD any points
B, C, D. and join BC, CD, DB: Then, becauſe the ſolid
angle at B is contained by the three plain angles CBA, ABD,
DBC, any two of them are greater ᵃ than the third; there-
fore the angles CBA, ABD are greater than the angle DBC:
For the ſame reaſon, the angles BCA, ACD are greater than
the angle DCB; and the angles CDA, ADB greater than
BDC: Wherefore the ſix angles CBA, ABD, BCA, ACD,
CDA, ADB are greater than the
three angles DBC, BCD, CDB: But
the three angles DBC, BCD, CDB
are equal to two right angles ᵇ: There-
fore the ſix angles CBA, ABD, BCA,
ACD, CDA, ADB are greater than
two right angles: And becauſe the
three angles of each of the triangles
ABC, ACD, ADB are equal to two
right angles, therefore the nine angles of theſe three triangles,
viz. the angles CBA, BAC, ACB, ACD, CDA, DAC, ADB,
DBA, BAD are equal to ſix right angles: Of theſe the ſix an-
gles CBA, ACB, ACD, CDA, ADB, DBA are greater than
two right angles: Therefore the remaining three angles BAC,
DAC, BAD, which contain the ſolid angle at A, are leſs than
four right angles.
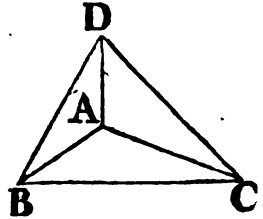
Next, Let the ſolid angle at A be contained by any number
of plane angles BAC, CAD, DAE, EAF, FAB; theſe toge-
ther are leſs than four right angles.

Let the planes in which the angles are, be cut by a plane, and
let the common ſections of it with thoſe
planes be BC, CD, DE, EF, FB: And
becauſe the ſold angle at B is contain-
ed by three plane angles CBA, ABF,
FBC, of which any two are greater ᵃ
than the third, the angles CBA, ABF
are greater than the angle FBC: For
the ſame reaſon, the two plane angles
at each of the points C, D, E, F,
viz. the angles which are at the baſes
of the triangles having the common
vertex A, are greater than the third
angle at the ſame point, which is one
of the angles of the polygon BCDEF:
Therefore all the angles at the baſes of the triangles are toge-
ther

ᵃ 20. 11.

ᵇ 32. 1.
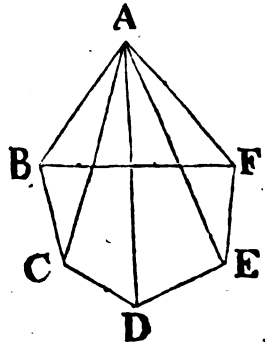
ther greater than all the angles of the polygon: And becaufe
all the angles of the triangles are together equal to twice as
many right angles as there are triangles [b]; that is, as there are b 32. 1.
fides in the polygon BCDEF: and that all the angles of the
polygon, together with four right angles, are likewife equal to
twice as many right angles as there are fides in the polygon [c]; c 1. Cor.
therefore all the angles of the triangles are equal to all the an- 32. 1.
gles of the polygon together with four right angles. But all
the angles at the bafes of the triangles are greater than all the
angles of the polygon, as has been proved. Wherefore the re-
maining angles of the triangles, viz. thofe at the vertex, which
contain the folid angle at A, are lefs than four right angles.
Therefore every folid angle, &c. Q. E. D.

PROP. XXII. THEOR.

IF every two of three plain angles be greater than the
third, and if the ftraight lines which contain them be
all equal; a triangle may be made of the ftraight lines
that join the extremities of thofe equal ftraight lines.

Let ABC, DEF, GHK be three plane angles, whereof e-
very two are greater than the third, and are contained by the
equal ftraight lines AB, BC, DE, EF, GH, HK; if their
extremities be joined by the ftraight lines AC, DF, GK, a
triangle may be made of three ftraight lines equal to AC, DF,
GK; that is, every two of them are together greater than the
third.
If the angles at B, E, H are equal; AC, DF, GK are al-
fo equal [a], and any two of them greater than the third: But a 4. 1.
if the angles are not all equal, let the angle ABC be not lefs
than either of the two at E, H; therefore the ftraight line
AC is not lefs than either of the other two DF, GK [b]; and b 4. or 24. 1.
it is plain that AC, together with either of the other two, muft
be greater than the third: Alfo DF with GK are greater than
AC: For, at the point B in the ftraight line AB make [c] the c 23. 1.

O 4   angle

angle ABL equal to the angle GHK, and make BL equal to
one of the ſtraight lines AB, BC, DE, EF, GH HK, and
join AL, LC; Then becauſe AB, BL are equal to GH, HK,
and the angle ABL to the angle GHK, the baſe AL is equal
to the baſe GK: And becauſe the angles at E, H are greater
than the angle ABC, of which the angle at H is equal to ABL;
therefore the remaining angle at E is greater than the angle LBC:
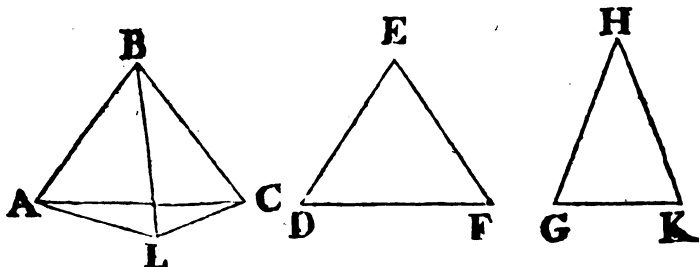
And becauſe the two ſides LB, BC are equal to the two DE,
EF, and that the angle DEF is greater than the angle LBC,
d 24. 1. the baſe DF is greater ᵇ than the baſe LC: And it has been
proved that GK is equal to AL; therefore DF and GK are
e 20. 1. greater than AL and LC: But AL and LC are greater ᵉ than
AC; much more than are DF and GK greater than AC. Where-
fore every two of theſe ſtraight lines AC, DF, GK are greater
f 22. 1. than the third; and, therefore, a triangle may be made ᶠ, the
ſides of which ſhall be equal to AC, DF, GK.　Q. E. D.

P R O P. XXIII.　P R O B.

See N.　**T**O make a ſolid angle which ſhall be contained by
three given plane angles, any two of them being
greater than the third, and all three together leſs than
four right angles.

Let the three given plane angles be ABC, DEF, GHK, any
two of which are greater than the third, and all of them toge-
ther leſs than four right angles. It is required to make a ſolid
angle contained by three plane angles equal to ABC, DEF,
GHK, each to each.

From

From the ſtraight lines containing the angles, cut off AB, BC, DE, EF, GH, HK all equal to one another ; and join AC, DF, GK: Then a triangle may be made a of three ſtraight   a 22. 11.
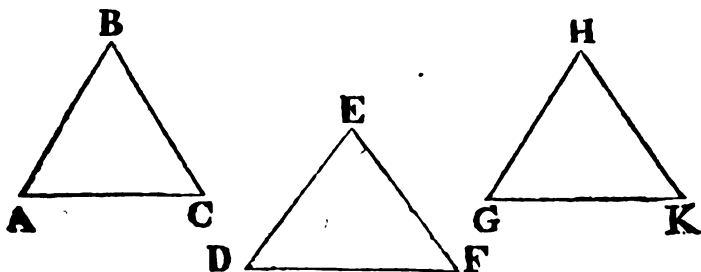
lines equal to AC, DF, GK.   Let this be the triangle LMN b,   b 22. 18.
ſo that AC be equal to LM, DF to MN, and GK to LN; and
about the triangle LMN deſcribe c a circle, and find its centre   c 5. 4.
X, which will either be within the triangle, or in one of its
ſides, or without it.

Firſt, Let the centre X be within the triangle, and join
LX, MX, NX: AB is greater than LX: If not, AB muſt ei-
ther be equal to, or leſs than LX; firſt, let it be equal : Then
becauſe AB is equal to LX, and that AB is alſo equal to BC,
and LX to XM, AB and BC are equal to LX and XM, each
to each; and the baſe AC is, by conſtruction, equal to the
baſe LM ; wherefore the angle ABC is equal to the angle
LXM d : For the ſame reaſon, the angle DEF is equal to the   d 8. 1.
angle MXN, and the angle GHK
to the angle NXL : Therefore the
three angles ABC, DEF, GHK
are equal to the three angles LXM,
MXN, NXL : But the three angles
LXM, MXN, NXL are equal to
four right angles e ; therefore alſo   e 2. Cor.
the three angles ABC, DEF, GHK   15. 1.
are equal to four right angles : But,
by the hypotheſis, they are leſs
than four right angles ; which is
abſurd ; therefore AB is not equal
to LX : But neither can AB be
leſs than LX : For, if poſſible, let it be leſs, and upon the
ſtraight line LM, on the ſide of it on which is the centre X,
deſcribe the triangle LOM. the ſides LO, OM, of which are
equal to AB, BC ; and becauſe the baſe LM is equal to the
baſe

bafe AC, the angle LOM is equal to the angle ABC[d] : And AB, that is, LO, by the hypothefis, is lefs than LX ; where-fore LO, OM fall within the triangle LXM ; for, if they fell upon its fides, or without it, they would be equal to, or greater than LX, XM[f] : Therefore the angle LOM, that is, the angle ABC, is greater than the angle LXM[f] : In the fame manner it may be proved that the angle DEF is greater than the angle MXN, and the angle GHK greater than the angle NXL: Therefore the three angles ABC, DEF, GHK are greater than the three angles LXM, MXF, NXL ; that is, than four right angles : But the fame angles ABC, DEF, GHK are lefs than four right angles ; which is abfurd : Therefore AB is not lefs than LX, and it has been proved that it is not equal to LX ; wherefore AB is greater than LX.
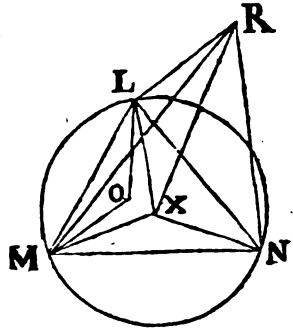
Next, Let the centre X of the circle fall in one of the fides of the triangle, viz. in MN, and join XL : In this cafe alfo AB is greater than LX. If not, AB is either equal to LX, or lefs than it : Firft, let it be equal to LX : There-fore AB and BC, that is, DE and EF, are equal to MX and XL, that is, to MN : But, by the conftruction, MN is equal to DF ; therefore DE, EF are equal to DF, which is im-poffible † : Wherefore AB is not e-qual to LX ; nor is it lefs ; for then, much more, an abfurdity would follow : Therefore AB is greater than LX.
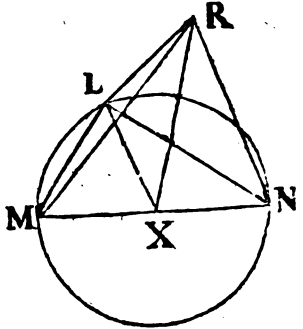
But, let the centre X of the circle fall within the triangle LMN, and join LX, MX, NX. In this cafe likewife AB is greater than LX: If not, it is either equal to, or lefs than LX : Firft, let it be equal ; it may be proved in the fame manner, as in the firft cafe, that the angle ABC is equal to the angle MXL, and GHK to LXN ; therefore the whole angle MXN is equal to the two angles, ABC, GHK : But ABC and GHK are together greater than the angle DEF ; therefore alfo the angle MXN is greater than DEF. And becaufe DE, EF

EF are equal to MX, XN, and the bafe DF to the bafe MN, the angle MXN is equal [d] to the angle DEF: And it has been proved that it is greater than DEF, which is abfurd. Therefore AB is not equal to LX. Nor yet is it lefs; for then, as has been proved in the firft cafe, the angle ABC is greater than the angle MXL, and the angle GHK greater than the angle LXN. At the point B in the ftraight line CB make the angle CBP equal to the angle GHK, and make BP equal to
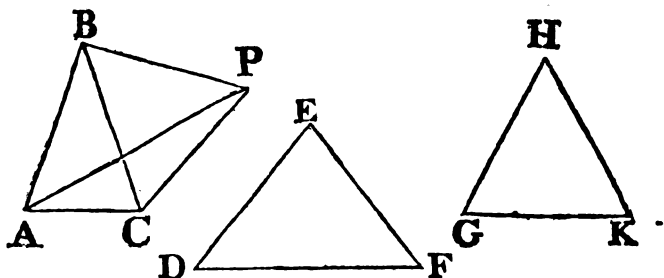
HK, and join CP, AP. And becaufe CB is equal to GH; CB, BP are equal to GH, HK, each to each, and they contain equal angles; wherefore the bafe CP is equal to the bafe GK, that is, to LN. And in the ifofceles triangles ABC, MXL, becaufe the angle ABC is greater than the angle MXL, therefore the angle MLX at the bafe is greater [g] than the angle ACB at the bafe. For the fame reafon, becaufe the angle GHK, or CBP, is greater than the angle LXN, the angle XLN is greater than the angle BCP. Therefore the whole angle MLN is greater than the whole angle ACP. And becaufe ML, LN are equal to AC, CP, each to each, but the angle MLN is greater than the angle ACP, the bafe MN is greater [h] than the bafe AP. And MN is equal to DF; therefore alfo DF is greater than AP. Again, becaufe DE, EF are equal to AB, BP, but the bafe DF greater than the bafe AP, the angle DEF is greater [k] than the angle ABP. And ABP is equal to the two angles ABC, CBP, that is, to the two angles ABC, GHK; therefore the angle DEF is greater than the two angles ABC, GHK; but it is alfo lefs than thefe; which is impoffible. Therefore AB is not lefs than LX;
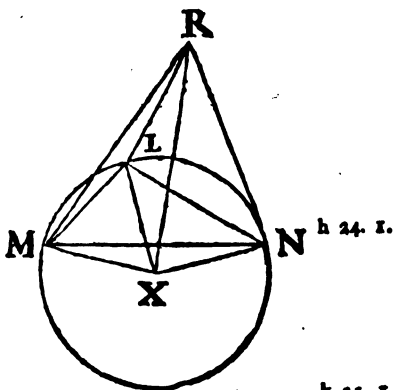
LX; and it has been proved that it is not equal to it; therefore AB is greater than LX.

a 12. 11.　　From the point X erect[a] XR at right angles to the plane of the circle LMN. And becaufe it has been proved in all the cafes, that AB is greater than LX, find a fquare equal to the excefs of the fquare of AB above

the fquare of LX, and make RX equal to its fide, and join RL, RM, RN. Becaufe RX is perpendicular to the plane of the circle LMN, it
b 3. def. 11. is[b] perpendicular to each of the ftraight lines LX, MX, NX. And becaufe LX is equal to MX, and XR common, and at right angles to each of them, the bafe RL is e-qual to the bafe RM. For the fame reafon, RN is equal to each of the two RL, RM. Therefore the three ftraight lines RL, RM, RN are all equal. And becaufe the fquare of XR is equal to the excefs of the fquare of AB above the fquare of LX; therefore the fquare of AB is equal to the fquares of
c 47. 1. LX, XR. But the fquare of RL is equal[c] to the fame fquares, becaufe LXR is a right angle. Therefore the fquare of AB is equal to the fquare of RL, and the ftraight line AB to RL. But each of the ftraight lines BC, DE, EF, GH, HK is equal to AB, and each of the two RM, RN is equal to RL. Where-fore AB, BC, DE, EF, GH, HK are each of them equal to each of the ftraight lines RL, RM, RN. And becaufe RL, RM, are equal to AB, BC, and the bafe LM to the bafe AC;
d 8. 1. the angle LRM is equal[d] to the angle ABC. For the fame reafon, the angle MRN is equal to the angle DEF, and NRL to GHK. Therefore there is made a folid angle at R, which is contained by three plane angles LRM, MRN, NRL, which are equal to the three given plane angles ABC, DEF, GHK, each to each. Which was to be done.
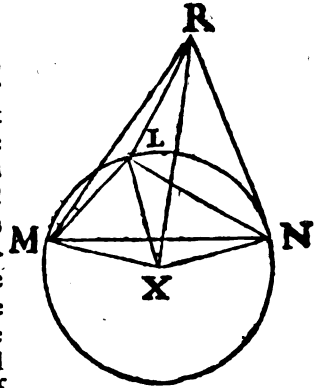
P R O P. A. T H E O R.

IF each of two folid angles be contained by three plane
angles equal to one another, each to each; the planes
in which the equal angles are, have the fame inclination
to one another.

Let there be two folid angles at the points A, B; and let
the angle at A be contained by the three plane angles CAD,
CAE, EAD; and the angle at B. by the three plane angles
FBG, FBH, HBG; of which the angle CAD is equal to the
angle FBG, and CAE to FBH, and EAD to HBG: The planes
in which the equal angles are, have the fame inclination to one
another.

In the ftraight line AC take any point K, and in the plane
CAD from K draw the ftraight line KD at right angles
to AC, and in the
plane CAE the ftraight
line KL at right angles
to the fame AC:
Therefore the angle
DKL is the inclina-
tion [a] of the plane
CAD to the plane
CAE: In BF take
BM equal to AK, and
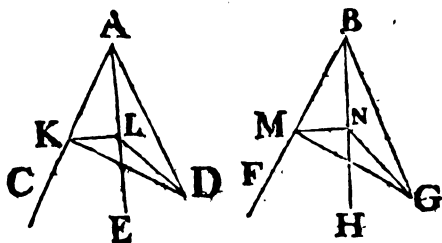
a 6. def. 11.

from the point M draw, in the planes FBG, FBH, the ftraight
lines MG, MN at right angles to BF; therefore the angle GMN
is the inclination [a] of the plane FBG to the plane FBH: Join
LD, NG; and becaufe in the triangles KAD, MBG, the
angles KAD, MBG are equal, as alfo the right angles AKD,
BMG, and that the fides AK, BM, adjacent to the equal
angles, are equal to one another, therefore KD is equal [b] to
MG, and AD to BG: For the fame reafon, in the triangles
KAL, MBN, KL is equal to MN, and AL to BN: And in
the triangles LAD, NBG, LA, AD are equal to NB, BG,
and they contain equal angles; therefore the bafe LD is e-
qual [c] to the bafe NG. Laftly, in the triangles KLD, MNG,
the fides DK, KL are equal to GM, MN, and the bafe LD to
the bafe NG; therefore the angle DKL is equal [d] to the angle
GMN: But the angle DKL is the inclination of the plane
CAD to the plane CAE, and the angle GMN is the inclina-

b 26. 1.

c 4. 1.

d 8. 1.

tion

tion of the plane FBG to the plane FBH, which planes have
therefore the fame inclination [a] to one another: And in the
fame manner it may be demonftrated, that the other planes in
which the equal angles are, have the fame inclination to one
another. Therefore, if two folid angles, &c. Q. E. D.

PROP. B. THEOR.

See N. . IF two folid angles be contained, each by three plane
angles which are equal to one another, each to each,
and alike fituated ; thefe folid angles are equal to one
another.

Let there be two folid angles at A and B, of which the folid
angle at A is contained by the three plane angles CAD, CAE,
EAD ; and that at B, by the three plane angles FBG, FBH,
HBG ; of which CAD is equal to FBG ; CAE to FBH ; and
EAD to HBG : The folid angle at A is equal to the folid angle
at B.
Let the folid angle at A be applied to the folid angle at B :
and, firft, the plane angle CAD being applied to the plane
angle FBG, fo as the point A may coincide with the point B,
and the ftraight line AC with BF ; then AD coincides with

BG, becaufe the angle CAD
is equal to the angle FBG :
And becaufe the inclination of
the plane CAE to the plane
CAD is equal [a] to the inclina-
tion of the plane FBH to the
plane FBG, the plane CAE
coincides with the plane FBH,
becaufe the planes CAD, FBG coincide with one another : And
becaufe the ftraight lines AC, BF coincide, and that the angle
CAE is equal to the angle FBH ; therefore AE coincides with
BH, and AD coincides with BG ; wherefore the plane EAD
coincides with the plane HBG : Therefore the folid angle A
coincides with the folid angle B, and confequently they are e-
qual [b] to one another. Q. E. D.
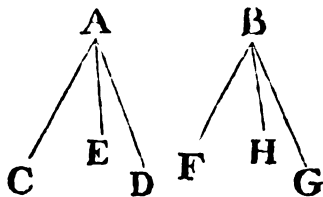
PROP.

PROP. C. THEOR.

SOLID figures contained by the fame number of e-qual and fimilar planes alike fituated, and having none of their folid angles contained by more than three plane angles ; are equal and fimilar to one another.

Let AG, KQ be two folid figures contained by the fame number of fimilar and equal planes, alike fituated, viz. let the plane AC be fimilar and equal to the plane KM; the plane AF to KP; BG to LQ; GD to QN; DE to NO; and laftly, FH fimilar and equal to PR : The folid figure AG is equal and fimilar to the folid figure KQ.

Becaufe the folid angle at A is contained by the three plane angles BAD, BAE, EAD, which, by the hypothefis, are e-qual to the plane angles LKN, LKO, OKN, which contain the folid angle at K, each to each ; therefore the folid angle at A is equal * to the folid angle at K : In the fame manner, the other folid angles of the figures are equal to one another. If, then, the folid figure AG be applied to the folid figure KQ,

firft, the plane fi-
gure AC being
applied to the
plane figure KM;
the ftraight line
A B coinciding
with KL, the fi-
gure AC muft
coincide with the
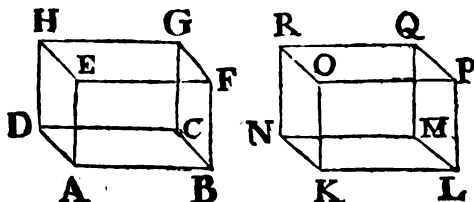
figure KM, becaufe they are equal and fimilar : Therefore the ftraight lines AD, DC, CB coincide with KN, NM, ML, each with each ; and the points A, D, C, B, with the points K, N, M, L : And the folid angle at A coincides with * the folid angle at K; wherefore the plane AF coincides with the plane KP, and the figure AF, with the figure KP, becaufe they are equal and fimilar to one another : Therefore the ftraight lines AE, EF, FB, coincide with KO, OP, PL ; and the points E, F, with the points O, P. In the fame manner, the figure AH coincides with the figure KR, and the ftraight line DH with NR, and the point H with the point R : And becaufe the folid angle at B is equal to the folid angle at L, it may be pro-ved, in the fame manner, that the figure BG coincides with
the

the figure LQ, and the ſtraight line CG with MQ, and the point G with the point Q: Since, therefore all the planes and ſides of the ſolid figure AG coincide with the planes and ſides of the ſolid figure KQ, AG is equal, and ſimilar to KQ: And, in the ſame manner, any other ſolid figures whatever contained by the ſame number of equal and ſimilar planes, alike ſituated, and having none of their ſolid angles contained by more than three plane angles, may be proved to be equal and ſimilar to one another. Q. E. D.

P R O P. XXIV. T H E O R.

See N.

IF a ſolid be contained by ſix planes, two and two of which are parallel; the oppoſite planes are ſimilar and equal parallelograms.

Let the ſolid CDGH be contained by the parallel planes AC, GF; BG, CE; FB, AE: Its oppoſite planes are ſimilar and equal parallelograms.

Becauſe the two parallel planes, BG, CE, are cut by the

a 16. 11. plane AC, their common ſections AB, CD, are parallel ᵃ. Again, becauſe the two parallel planes BF, AE, are cut by the plane AC, their common ſections AD, BC, are parallel ᵃ: And AB is parallel to CD; therefore AC is a parallelogram. In like manner, it may be proved that each of the figures CE, FG, GB, BF, AE is a parallelogram: Join AH, DF; and becauſe AB is parallel to DC, and BH to CF; the two ſtraight lines AB, BH, which meet one another, are parallel to DC and CF which meet one another, and are not in the ſame plane with the other two; wherefore they con-

b. 10. 11. tain equal angles ᵇ; the angle ABH is therefore equal to the angle DCF: And becauſe AB, BH, are equal to DC, CF, and the angle ABH equal to the angle DCF; therefore the baſe

c 4. 1. AH is equal ᶜ to the baſe DF, and the triangle ABH to the tri-

d 34. 1. angle DCF: And the parallelogram BG is double ᵈ of the triangle ABH, and the parallelogram CE double of the triangle DCF; therefore the parallelogram BG is equal and ſimilar to the parallelogram CE. In the ſame manner it may be proved, that the parallelogram AC is equal and ſi-

milar
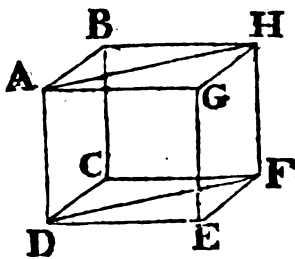
milar to the parallelogram GF, and the parallelogram AE to
BF. Therefore, if a folid, &c. Q. E. D.

PROP. XXV. THEOR.

IF a folid parallelepiped be cut by a plane parallel to two
of its oppofite planes; it divides the whole into two
folids, the bafe of one of which fhall be to the bafe of
the other, as the one folid is to the other.

Let the folid parallelepiped ABCD be cut by the plane EV,
which is parallel to the oppofite planes AR, HD, and divides
the whole into the two folids ABFV, EGCD; as the bafe
AEFY of the firft is to the bafe EHCF of the other, fo is the
folid ABFV to the folid EGCD.

Produce AH both ways, and take any number of ftraight
lines HM, MN, each equal to EH, and any number AK, KL,
each equal to EA, and complete the parallelograms LO, KY,
HQ, MS, and the folids LP, KR, HU, MT: Then, becaufe
the ftraight lines LK, KA, AE are all equal, the parallelograms
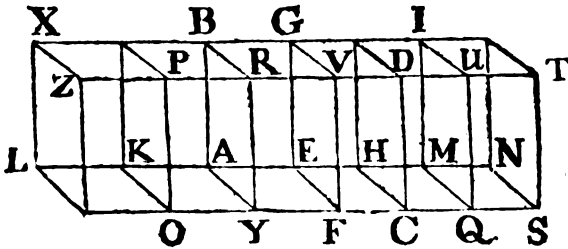
LO, KY, AF are equal [a]: And likewife the parallelograms KX, a 36. 1.
KB, AG [a]; as alfo [b] the parallelograms LZ, KP, AR, becaufe b 24. 11.
they are oppofite planes: For the fame reafon, the paral-
lelograms EC, HQ, MS are equal [a]; and the parallelograms
HG, HI, IN, as alfo [b] HD, MU, NT: Therefore three planes
of the folid LP, are equal and fimilar to three planes of the fo-
lid KR, as alfo to three planes of the folid AV: But the three
planes oppofite to thefe three are equal and fimilar [b] to them
in the feveral folids, and none of their folid angles are contain-
ed by more than three plane angles: Therefore the three folids
LP, KR, AV are equal [c] to one another: For the fame reafon, c C. 11.
the three folids ED, HU, MT are equal to one another: There-
P fore

fore what multiple foever the bafe LF is of the bafe AF, the
fame multiple is the folid LV of the folid AV: For the fame
reafon, whatever multiple the bafe NF is of the bafe HF, the
fame multiple is the folid NV of the folid ED: And if the bafe
e C. 11.   LF be equal to the bafe NF, the folid LV is equal e to the fo-
lid NV; and if the bafe LF be greater than the bafe NF, the
folid LV is greater than the folid NV; and if lefs, lefs: Since
then there are four magnitudes, viz. the two bafes AF, FH,
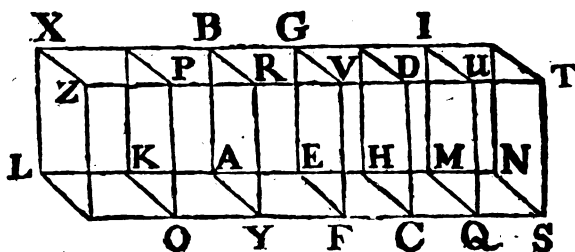
and the two folids AV, ED, and of the bafe AF and folid AV,
the bafe LF and folid LV are any equimultiples whatever; and
of the bafe FH and folid ED, the bafe FN and folid NV are a-
ny equimultiples whatever; and it has been proved, that if the
bafe LF is greater than the bafe FN, the folid LV is greater
than the folid NV; and if equal, equal; and if lefs, lefs.
d 5. def. 5.   Therefore d as the bafe AF is to the bafe FH, fo is the folid AV
to the folid ED. Wherefore, if a folid, &c. Q. E. D.

PROP. XXVI. PROB.

See N.   AT a given point in a given ftraight line, to make a
folid angle equal to a given folid angle contained
by three plane angles.

Let AB be a given ftraight line, A a given point in it, and
D a given folid angle contained by the three plane angles EDC,
EDF, FDC: It is required to make at the point A in the ftraight
line AB a folid angle equal to the folid angle D.

In the ftraight line DF take any point F, from which draw
a 11. 11.   FG perpendicular to the plane EDC, meeting that plane in
G; join DG, and at the point A in the ftraight line AB
b 23. 1.   make b the angle BAL equal to the angle EDC, and in the
plane BAL make the angle BAK equal to the angle EDG;
c 12. 11.   then make AK equal to DG, and from the point K erect c KH
                                                               at

at right angles to the plane BAL; and make KH equal to GF, and join AH: Then the folid angle at A, which is contained by the three plane angles BAL, BAH, HAL, is equal to the folid angle at D contained by the three plane angles EDC, EDF, FDC.

Take the equal ftraight lines AB, DE, and join HB, KB, FE, GE: And becaufe FG is perpendicular to the plane EDC, it makes right angles d with every ftraight line meeting it in that plane: Therefore each of the angles FGD, FGE is a right angle: For the fame reafon, HKA HKB are right angles: And becaufe KA, AB are equal to GD, DE, each to each, and contain equal angles, therefore the bafe BK is equal e to the bafe EG: And KH is equal to GF, and HKB, FGE, are right angles, therefore HB is equal e to FE: Again, becaufe AK, KH are equal to DG, GF, and contain right angles, the bafe AH is equal to the bafe DF; and AB is equal to DE; therefore HA, AB are equal to FD, DE, and the bafe HB is equal to the bafe FE, therefore the angle BAH is equal f to the angle EDF: For the fame reafon, the angle HAL is equal to the angle FDC. Becaufe if AL and DC be made equal, and KL, HL, GC,
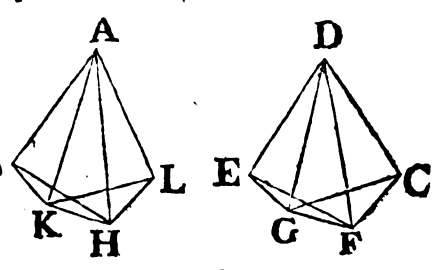
FC be joined, fince the whole angle BAL is equal to the whole EDC, and the parts of them BAK, EDG are, by the conftruction, equal; therefore the remaining angle KAL is equal to the remaining angle GDC: And becaufe KA, AL are equal to GD, DC, and contain equal angles, the bafe KL is equal e to the bafe GC: and KH is equal to GF, fo that LK, KH are equal to CG, GF, and they contain right angles; therefore the bafe HL is equal to the bafe FC: Again, becaufe HA, AL are equal to FD, DC, and the bafe HL to the bafe FC, the angle HAL is equal f to the angle FDC: Therefore, becaufe the three plane angles BAL, BAH, HAL, which contain the folid angle at A, are equal to the three plane angles EDC, EDF, FDC, which contain the fold angle at D, each to each, and are fituated in the fame order, the folid angle at A is equal g to the folid angle at D Therefore, at a given point in a given ftraight line, a folid angle has been made equal to a given folid angle contained by three plane angles. Which was to be done.

PROP.

PROP. XXVII. PROB.

TO defcribe from a given ftraight line a folid paralle-
lepiped fimilar, and fimilarly fituated to one given.

Let AB be the given ftraight line, and CD the given folid pa-
rallelepiped. It is required from AB to defcribe a folid paralle-
lepiped fimilar, and fimilarly fituated to CD.

a 26. 11.   At the point A of the given ftraight line AB, make a folid
angle equal to the folid angle at C, and let BAK, KAH, HAB
be the three plane angles which contain it, fo that BAK be e-
qual to the angle ECG, and KAH to GCF, and HAB to
b 12. 6.   FCE : And as EC to CG, fo make b BA to AK; and as GC to
c 22. 5.   CF, fo make b KA to AH; wherefore, ex aequali c, as EC to
CF, fo is BA to AH : Complete the parallelogram BH, and

the folid AL : And
becaufe, as EC to
CG, fo BA to AK,
the fides about the
equal angles ECG,
BAK are proporti-
onals; therefore the
parallelogram   BK
is fimilar to EG.
For the fame rea-
fon, the   parallelo-
gram KH is fimilar to GF, and HB to FE.   Wherefore three
parallelograms of the folid AL are fimilar to three of the folid
CD; and the three oppofite ones in each folid are equal d and
d 24. 11.   fimilar to thefe, each to each.   Alfo, becaufe the plane angles
which contain the folid angles of the figures are equal, each to
each, and fituated in the .fame order, the folid angles are e-
e B. 11.   qual e, each to each.   Therefore the folid AL is fimilar f to the
f 11.def.11. folid CD.   Wherefore from a given ftraight line AB a folid p -
rallelepiped AL has been defcribed fimilar, and fimilarly fituated
to the given one CD. Which was to be done.
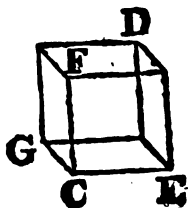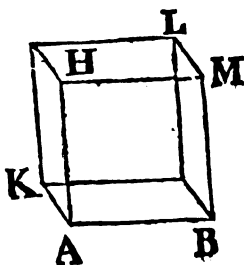
PROP. XXVIII. THEOR.

IF a folid parallelepiped be cut by a plane paffing through
the diagonals of two of the oppofite planes ; it fhall
be cut in two equal parts.

Let AB be a folid parallelepiped, and DE, CF, the diago-
nals of the oppofite parallelograms AH, GB, viz. thofe which
are drawn betwixt the equal angles in each : And becaufe CD,
FE are each of them parallel to GA, and not in the fame plane
with it, CD, FF are parallel ª ; wherefore the diagonals CF, ª 9, 11¡
DE are in the plane in which the pa-
rallels are, and are themfelves paral-
lels ᵇ : And the plane CDEF fhall cut
the folid AB into two equal parts.

ᵇ 16. 11.
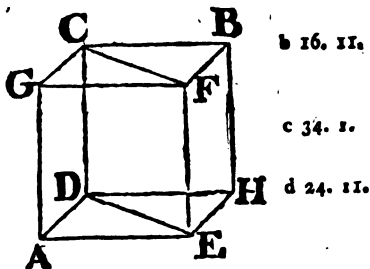
Becaufe the triangle CGF is equal
ᶜ to the triangle CBF, and the triangle
DAE to DHE ; and that the paral-
lelogram CA is equal ᵈ and fimilar to
the oppofite one BE ; and the paral-
lelogram GE to CH : Therefore the
prifm contained by the two triangles
CGF, DAE, and the three parallelograms CA, GE, EC, is
equal ᵉ to the prifm contained by the two triangles CBF, DHE, ᵉ C. 11.
and the three parallelograms BE, CH, EC ; becaufe they are
contained by the fame number of equal and fimilar planes, alike
fituated, and none of their folid angles are contained by more
than three plane angles. Therefore the folid AB is cut into two
equal parts by the plane CDEF. Q. E. D.

ᶜ 34. 1.

ᵈ 24. 11.

' N. B The infifting ftraight lines of a parallelepiped, men-
' tioned in the next and fome following propofitions, are the
' fides of the parallelograms betwixt the bafe and the oppofite
' plane parallel to it.'

PROP. XXIX. THEOR.

SOLID parallelepipeds upon the fame bafe, and of the
fame altitude, the infifting ftraight lines of which are
terminated in the fame ftraight lines in the plane oppofite
to the bafe, are equal to one another.

P 4

Let

Let the folid parallelepipeds AH, AK be upon the fame bafe AB, and of the fame altitude, and let their infifting ftraight lines AF, AG, LM, LN, be terminated in the fame ftráight line FN, and CD, CE, BH, BK be terminated in the fame ftraight line DK; the folid AH is equal to the folid AK.

Firft, Let the parallelograms DG, HN, which are oppofite to the bafe AB, have a common fide HG: Then, becaufe the folid AH is cut by the plane AGHC paffing through the diago-nals AG, CH of, the oppofite planes ALGF, CBHD, AH is cut into two equal parts [a] by the plane AGHC: Therefore the folid AH is double of the prifm which is contained be-twixt the triangles ALG, CBH: For the fame reafon, becaufe the folid AK is cut by the plane LGHB through the diagonals LG, BH of the oppofite planes ALNG, CBKH, the folid AK is double of the fame prifm which is contained betwixt the triangles ALG, CBH. Therefore the folid AH is equal to the folid AK.
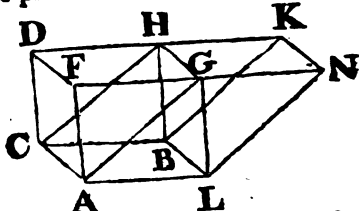
But, let the parallelograms DM, EN oppofite to the bafe, have no common fide: Then, becaufe CH, CK are parallelo-grams, CB is equal [b] to each of the oppofite fides DH, EK; wherefore DH is equal to EK: Add, or take away the common part HE; then DE is equal to HK: Wherefore alfo the tri-angle CDE is equal [c] to the triangle BHK: And the parallelo-gram DG is equal [d] to the parallelogram HN: For the fame reafon, the triangle AFG is equal to the triangle LMN, and the parallelogram CF is equal [e] to the parallelogram BM, and
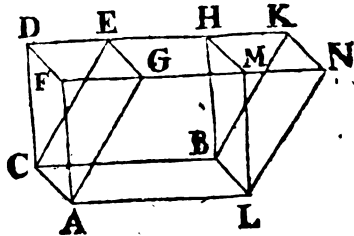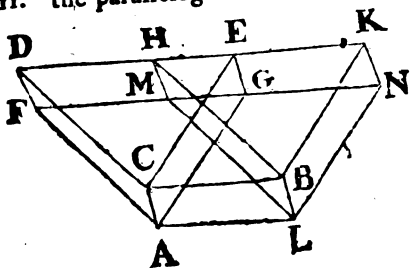
CG to BN; for they are oppofite. Therefore the prifm which is contained by the two triangles AFG, CDE, and the three parallelograms AD, DG, GC is equal [f] to the prifm, contain-ed by the two triangles LMN, BHK, and the three parallelo-grams BM, MK, KL. If therefore the prifm LMNBHK be taken

taken from the folid of which the bafe is the parallelogram
AB, and in which FDKN is the one oppofite to it; and if
from this fame folid there be taken the prifm AFGCDE;
the remaining folid, viz. the parallelepiped AH, is equal to the
remaining parallelepiped AK. Therefore folid parallelepipeds,
&c. Q. E. D.

PROP. XXX. THEOR.

SOLID parallelepipeds upon the fame bafe, and of the See N.
fame altitude, the infifting ftraight lines of which are
not terminated in the fame ftraight lines in the plane
oppofite to the bafe, are equal to one another.

Let the parallelepipeds CM, CN be upon the fame bafe AB,
and of the fame altitude, but their infifting ftraight lines AF,
AG, LM, LN, CD, CE, BH, BK not terminated in the fame
ftraight lines: The folids CM, CN are equal to one another.
Produce FD, MH, and NG, KE, and let them meet one
another in the points O, P, Q, R; and join AO, LP, BQ,
CR: And becaufe the plane LBHM is parallel to the oppofite
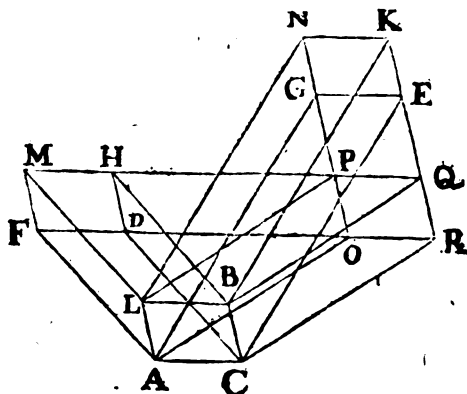
plane ACDF, and that the plane LBHM is that in which are
the parallels LB, MHPQ, in which alfo is the figure BLPQ;
and the plane ACDF is that in which are the parallels AC,
FDOR, in which alfo is the figure CAOR; therefore the fi-
gures BLPQ, CAOR are in parallel planes: In like manner,
becaufe the plane ALNG is parallel to the oppofite plane CBKE,
and that the plane ALNG is that in which are the parallels

AL,

AL, OPGN, in which alfo is the figure ALPO; and the plane
CBKE is that in which are the parallels CB, RQEK, in which
alfo is the figure CBQR; therefore the figures ALPO, CBQR
are in parallel planes: And the planes ACBL, ORQP are pa-
rallel; therefore the folid CP is a parallelepiped: But the fo-
lid CM, of which the bafe is ACBL, to which FDHM is the
oppofite parallelogram, is equal[a] to the folid CP, of which the
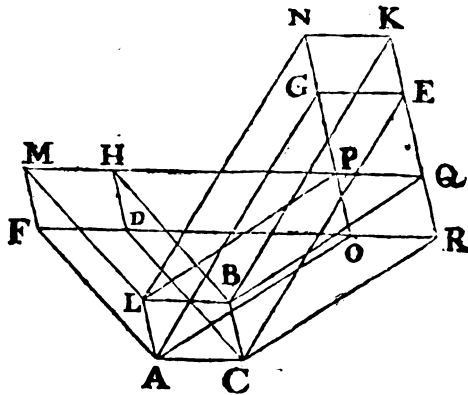
bafe is the parallelogram ACBL, to which ORQP is the one
oppofite; becaufe they are upon the fame bafe, and their in-
fifting ftraight lines AF, AO, CD, CR; LM, LP, BH, BQ
are in the fame ftraight lines FR, MQ: And the folid CP is e-
qual[a] to the folid CN; for they are upon the fame bafe ACBL,
and their infifting ftraight lines AO, AG, LP, LN; CR, CE,
BQ, BK are in the fame ftraight lines ON, RK: Therefore the
folid CM is equal to the folid CN. Wherefore folid parallele-
pipeds, &c. Q. E. D.

P R O P. XXXI. T H E O R.

SOLID parallelepipeds which are upon equal bafes, and
of the fame altitude, are equal to one another.

Let the folid parallelepipeds AE, CF, be upon equal bafes
AB, CD, and be of the fame altitude; the folid AE is equal
to the folid CF.

Firft, Let the infifting ftraight lines be at right angles to the
bafes AB, CD, and let the bafes be placed in the fame plane,
and

and fo as that the fides CL, LB be in a ftraight line ; there-
fore the ftraight line LM, which is at right angles to the plane
in which the bafes are, in the point L, is common ^{a} to the two ^{a} 15. 11;
folids AE, CF ; let the other infifting lines of the folids be
AG, HK, BE ; DF, OP, CN :   And firft, let the angle ALB be
equal to the angle CLD ; then AL, LD are in a ftraight line^b. ^{b} 14. 1.
Produce OD, HB, and let them meet in Q, and complete the
folid parallelepiped LR, the bafe of which is the parallelogram
LQ, and of which LM is one of its infifting ftraight lines :
Therefore, becaufe the parallelogram AB is equal to CD, as the
bafe AB is to the bafe LQ, fo is ^{c} the bafe CD to the fame ^{c} 7. 5.
LQ :   And becaufe the folid parallelepiped AR is cut by the
plane LMEB, which is parallel to the oppofite planes AK, DR ;
as the bafe AB is to the bafe LQ, fo is ^{d} the folid AE to the ^{d} 25. 11.
folid LR : For the fame reafon, becaufe the folid parallelepiped
CR is cut by the plane LMFD, which is parallel to the oppofite
planes CP, BR; as
the bafe CD to the
bafe LQ, fo is the
folid CF to the fo-
lid LR : But as the
bafe AB to the bafe
LQ, fo the bafe CD
to the bafe LQ, as
before was proved :
Therefore as the fo-
lid AE to the folid
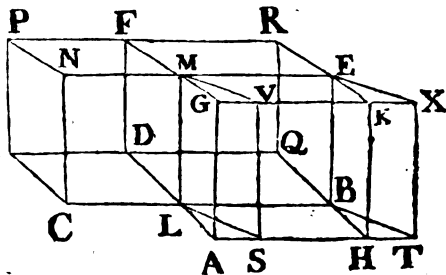
LR, fo is the folid CF to the folid LR ; and therefore the folid
AE is equal ^{e} to the folid CF. ^{e} 9. 5.
    But let the folid parallelepipeds SE, CF be upon equal bafes
SB, CD, and be of the fame altitude, and let their infifting
ftraight lines be at right angles to the bafes ; and place the
bafes SB, CD in the fame plane, fo that CL, LB be in a ftraight
line ; and let the angles SLB, CLD be unequal ; the folid SE
is alfo in this cafe equal to the folid CF : Produce DL, TS un-
til they meet in A, and from B draw BH parallel to DA ; and
let HB, OD produced meet in Q, and complete the folids AE,
LR : Therefore the folid AE, of which the bafe is the parallelo-
gram LE, and AK the one oppofite to it, is equal ^{f} to the fo- ^{f} 29. 11.
lid SE, of which the bafe is LE, and to which SX is oppofite ;
for they are upon the fame bafe LE, and of the fame altitude,
and their infifting ftraight lines, viz. LA, LS, BH, BT ; MG,
MV, EK, EX are in the fame ftraight lines AT, GX : And be-
                                                              caufe

caufe the parallelogram AB is equal ^s to SB, for they are upon the fame bafe LB, and between the fame parallels LB, AT; and that the bafe SB is equal to the bafe CD; therefore the bafe AB is equal to the bafe CD, and the angle ALB is equal to the angle CLD: Therefore, by the firft cafe, the folid AE is equal to the folid CF; but the folid AE is equal to the folid SE, as was demonftrated; therefore the folid SE is equal to the folid CF.
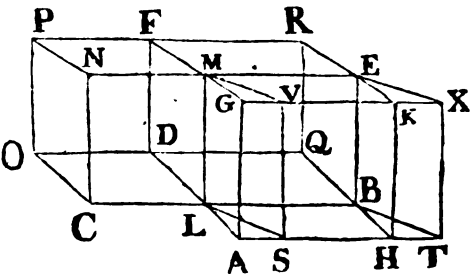
But, if the infifting ftraight lines AG, HK, BE, LM; CN, RS, DF, OP, be not at right angles to the bafes AB, CD; in this cafe likewife the folid AE is equal to the folid CF: From the points G, K, E, M; N, S, F, P, draw the ftraight lines GQ, KT, EV, MX; NY, SZ, FI, PU, perpendicular ^h to the plane in which are the bafes AB, CD; and let them meet it in the points Q, T, V, X; Y, Z, I, U, and join QT, TV, VX, XQ; YZ, ZI, IU, UY: Then, becaufe GQ, KT, are at right
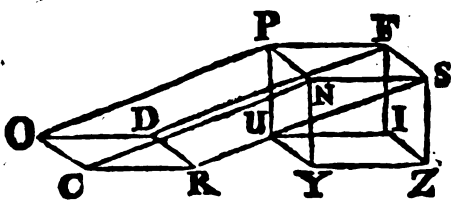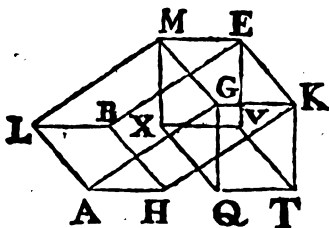
angles to the fame plane, they are parallel ⁱ to one another: And MG, EK are parallels; therefore the planes MQ, ET, of which one paffes through MG, GQ, and the other through EK, KT which are parallel to MG, GQ, and not in the fame plane with them, are parallel ^k to one another: For the fame reafon, the planes MV, GT are parallel to one another: Therefore the folid QE is a parallelepiped: In like manner, it may be proved, that the folid YF is a parallelepiped: But, from what has been demonftrated, the folid EQ is equal to the folid FY, becaufe they are upon equal bafes MK, PS, and of the fame altitude, and have their infifting ftraight lines at right angles
to

to the bafes: And the folid EQ is equal [1] to the folid AE; and Book XI.
the folid FY to the folid CF; becaufe they are upon the fame
bafes and of the fame altitude: Therefore the folid AE is equal | 29. or 30.
to the folid CF. Wherefore folid parallelepipeds, &c. Q. E. D.

PROP. XXXII. THEOR.

SOLID parallelepipeds which have the fame altitude, are See N.
to one another as their bafes.

Let AB, CD be folid parallelepipeds of the fame altitude:
They are to one another as their bafes; that is, as the bafe AE
to the bafe CF, fo is the folid AB to the folid CD.

To the ftraight line FG apply the parallelogram FH equal [a] a Cor.45.1.
to AE, fo that the angle FGH be equal to the angle LCG;
and complete the folid parallelepiped GK upon the bafe FH,
one of whofe infifting lines is FD, whereby the folids CD, GK
muft be of the fame altitude: Therefore the folid AB is equal [b] b 31. 11.
to the folid
GK, becaufe
they are upon
equal bafes
AE, FH, and
are of the fame
altitude: And
becaufe the fo-
lid parallelepi-
ped CK is cut
by the plane DG which is parallel to its oppofite planes, the bafe
HF is [c] to the bafe FC, as the folid HD to the folid DC: But c 25. 11.
the bafe HF is equal to the bafe AE, and the folid GK to the
folid AB: Therefore, as the bafe AE to the bafe CF, fo is the
folid AB to the folid CD. Wherefore folid parallelepipeds, &c.
Q. E. D.
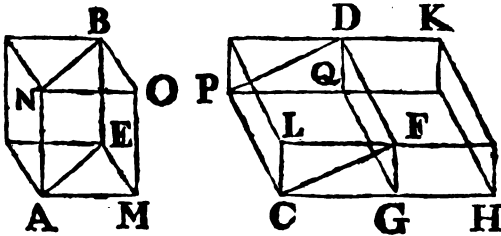
COR. From this it is manifeft that prifms upon triangular ba-
fes, of the fame altitude, are to one another as their bafes.

Let the prifms, the bafes of which are the triangles AEM,
CFG, and NBO, PDQ the triangles oppofite to them, have
the fame altitude; and complete the parallelograms AE, CF,
and the folid parallelepipeds AB, CD, in the firft of which let
MO, and in the other let GQ be one of the infifting lines. And
becaufe the folid parallelepipeds AB, CD have the fame alti-
tude, they are to one another as the bafe AE is to the bafe
CF;

CF; wherefore the prifms, which are their halves ^d, are to one another, as the bafe AE to the bafe CF; that is, as the triangle AEM to the triangle CFG.

PROP. XXXIII. THEOR.

SIMILAR folid parallelepipeds are one to another in the triplicate ratio of their homologous fides.

Let AB, CD be fimilar folid parallelepipeds, and the fide AE homologous to the fide CF: The folid AB has to the folid CD, the triplicate ratio of that which AE has to CF.

Produce AE, GE, HE, and in thefe produced take EK e-qual to CF, EL equal to FN, and EM equal to FR; and com-plete the parallelogram KL, and the folid KO: Becaufe KE, EL are equal to CF, FN, and the angle KEL equal to the angle CFN, becaufe it is equal to the angle AEG which is equal to CFN, by reafon that the folids AB, CD are fimilar; therefore the paral-lelogram KL is fimilar and equal to the parallelogram CN: For the fame reafon, the parallelogram MK is fimilar and equal to CR, and alfo OE to FD: There-fore three paralle-lograms of the fo lid KO are equal and fimilar to three parallelograms of the folid CD: And the three oppofite ones in each folid are equal ^a and fimilar to thefe:
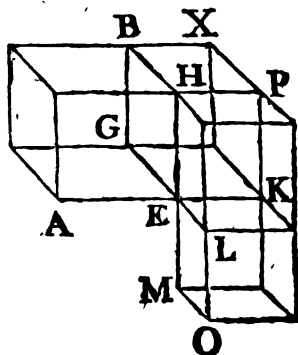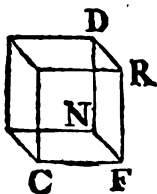
Therefore the fo-lid KO is equal ^b

and fimilar to the folid CD: Complete the parallelogram GK, and complete the folids EX, LP upon the bafes GK, KL, fo that EH be an infifting ftraight line in each of them, whereby they muft be of the fame altitude with the folid AB: And becaufe the folids AB, CD are fimilar, and, by permutation, as AE is to CF, fo is EG to FN, and fo is EH to FR; and FC is equal to EK, and FN to EL, and FR to EM; therefore as AE to EK, fo is EG to EL, and fo is HE to EM: But, as AE to EK, fo ^c is the parallelogram AG to the parallelogram GK; and as GE to EL, fo is ^c GK to KL;

and

and as HE to EM, fo ^{c} is PE to KM : Therefore as the parallelo-
gram AG to the parallelogram GK, fo is GK to KL, and PE
to KM : But as AG to GK, fo ^{d} is the folid AB to the folid c 1. 6.
d 25. 11.
EX; and as GK to KL, fo ^{d} is the folid EX to the folid PL ;
and as PE to KM, fo ^{d} is the folid PL to the folid KO : And
therefore as the folid AB to the folid EX, fo is EX to PL,
and PL to KO : But if four magnitudes be continual propor-
tionals, the firſt is ſaid to have to the fourth the triplicate ratio
of that which it has to the ſecond : Therefore the folid AB
has to the folid KO, the triplicate ratio of that which AB has
to EX : But as AB is to EX, fo is the parallelogram AG to
the parallelogram GK, and the ſtraight line AE to the ſtraight
line EK. Wherefore the folid AB has to the folid KO, the tri-
plicate ratio of that which AE has to EK. And the folid KO
is equal to the folid CD, and the ſtraight line EK is equal to
the ſtraight line CF. Therefore the folid AB has to the folid
CD, the triplicate ratio of that which the fide AE has to the
homologous fide CF, &c. Q. E. D.

COR. From this it is manifeſt, that, if four ſtraight lines be
continual proportionals, as the firſt is to the fourth, fo is the
folid parallelepiped deſcribed from the firſt to the fimilar folid
fimilarly deſcribed from the ſecond ; becauſe the firſt ſtraight
line has to the fourth the triplicate ratio of that which it has
to the ſecond.

PROP. D. THEOR.

SOLID parallelepipeds contained by parallelograms See N.
equiangular to one another, each to each, that is,
of which the folid angles are equal, each to each, have to
one another the ratio which is the fame with the ratio
compounded of the ratios of their fides.

Let AB, CD be folid parallelepipeds, of which AB is con-
tained by the parallelograms AE, AF, AG equiangular, each
to each, to the parallelograms CH, CK, CL which contain the
folid CD. The ratio which the folid AB has to the folid CD is
the fame with that which is compounded of the ratios of the
fides AM to DL, AN to DK, and AO to DH.

Prodúce

Produce MA, NA, OA to P, Q, R, fo that AP be equal to DL, AQ to DK, and AR to DH; and complete the folid parallelepiped AX contained by the parallelograms AS, AT, AV fimilar and equal to CH, CK, CL, each to each. There-
a C. 11. fore the folid AX is equal [a] to the folid CD. Complete likewife the folid AY, the bafe of which is AS, and of which AO is one of its infifting ftraight lines. Take any ftraight line a, and as MA to AP, fo make a to b; and as NA to AQ, fo make b to c; and as AO to AR, fo c to d: Then, becaufe the parallelogram AE is equiangular to AS, AE is to AS, as the ftraight line a to c, as is demonftrated in the 23. Prop. Book 6. and the folids AB, AY, being betwixt the parallel planes BOY, EAS, are of the fame altitude. Therefore the folid AB
b 32. 11. is to the folid AY, as [b] the bafe AE to the bafe AS; that is, as the ftraight line a is to c. And the folid AY is to the folid
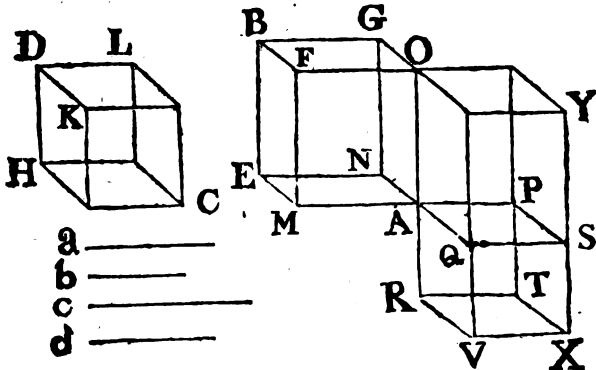
c 25. 11. AX, as [c] the bafe OQ is to the bafe QR; that is, as the ftraight line OA to AR; that is, as the ftraight line c to the ftraight line d. And becaufe the folid AB is to the folid AY, as a is to c, and the folid AY to the folid AX, as c is to d; ex aequali, the folid AB is to the folid AX, or CD which is equal to it, as the ftraight line a is to d. But the ratio of a to d is faid to
d def. A. 5. be compounded [d] of the ratios of a to b, b to c, and c to d, which are the fame with the ratios of the fides MA to AP, NA to AQ, and OA to AR, each to each. And the fides AP, AQ, AR are equal to the fides DL, DK, DH, each to each. There-fore the folid AB bas to the folid CD the ratio which is the fame with that which is compounded of the ratios of the fides AM to DL, AN to DK, and AO to DH. Q. E. D.

PROP.

PROP. XXXIV. THEOR.

THE bases and altitudes of equal solid parallelepi- See N.
peds, are reciprocally proportional; and if the
bases and altitudes be reciprocally proportional, the solid
parallelepids are equal.

Let AB, CD be equal solid parallelepipeds; their bases are
reciprocally proportional to their altitudes; that is, as the base
EH is to the base NP, so is the altitude of the solid CD to the
altitude of the solid AB.

First, Let the insisting straight lines AG, EF, LB, HK;
CM, NX, OD, PR be at right angles to the bases. As the base
EH to the base NP, so is
CM to AG. If the base
EH be equal to the base
NP, then because the so-
lid AB is likewise equal
to the solid CD, CM shall
be equal to AG. Because,
if the bases EH, NP be e-
qual, but the altitudes
AG, CM be not equal,
neither shall the solid AB be equal to the solid CD. But the
solids are equal, by the hypothesis. Therefore the altitude CM
is not unequal to the altitude AG; that is, they are equal.
Wherefore as the base EH to the base NP, so is CM to AG.
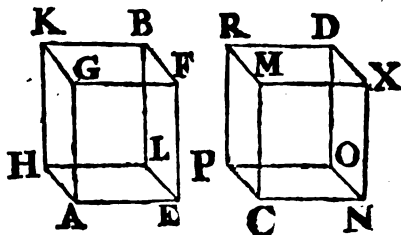
Next, Let the bases EH, NP not be equal, but EH greater
than the other: Since then the solid AB is equal to the solid
CD, CM is therefore
greater than AG: For,
if it be not, neither al-
so, in this case, would
the solids AB, CD be
equal, which, by the
hypothesis, are equal.
Make then CT equal to
AG, and complete the
solid parallelepiped CV
of which the base is
NP, and altitude CT.
Because the solid AB
is equal to the solid CD, therefore the solid AB is to the
solid
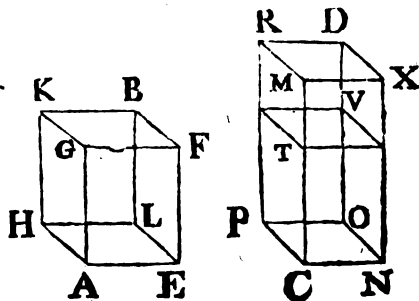
folid CV, as a the folid CD to the folid CV. But as the fo-
lid AB to the folid CV, fo b is the bafe EH to the bafe NP; for
the folids AB, CV are of the fame altitude; and as the folid
CD to CV, fo c is the bafe MP to the bafe PT, and fo d is
the ftraight line MC to CT; and CT is equal to AG. There-
fore, as the bafe EH to the bafe NP, fo is MC to AG. Where-
fore the bafes of the folid parallelepipeds AB, CD are recipro-
cally proportional to their altitudes.

Let now the bafes of the folid parallelepipeds AB, CD be re-
ciprocally proportional to their altitudes; viz. as the bafe EH
to the bafe NP, fo the al-
titude of the folid CD to
the altitude of the folid
AB; the folid AB is e-
qual to the folid CD. Let
the infifting lines be, as
before, at right angles to
the bafes. Then, if the
bafe EH be equal to the
bafe NP, fince EH is to

NP, as the altitude of the folid CD is to the altitude of the fo-
lid AB, therefore the altitude of CD is equal e to the altitude
of AB. But folid parallelepipeds upon equal bafes, and of the
fame altitude, are equal f to one another; therefore the folid AB
is equal to the folid CD.

But let the bafes EH, NP be unequal, and let EH be the
greater of the two. Therefore, fince as the bafe EH to the bafe
NP, fo is CM the alti-
tude of the folid CD to
AG the altitude of AB,
CM is greater e than
AG. Again, Take CT
equal to AG, and com-
plete, as before, the fo-
lid CV. And, becaufe
the bafe EH is to the
bafe NP, as CM to AG,
and that AG is equal
to CT, therefore the bafe

EH is to the bafe NP, as MC to CT. But as the bafe EH is to NP,
fo b is the folid AB to the folid CV; for the folids AB, CV are of
the fame altitude; and as MC to CT, fo is the bafe MP to the bafe
PT.

PT, and the folid CD to the folid ᶜ CV : And therefore as the
folid AB to the folid CV, fo is the folid CD to the folid CV ;
that is, each of the folids AB, CD has the fame ratio to the ᶜ 25. 11.
folid CV ; and therefore the folid AB is equal to the folid CD.

Second general cafe. Let the infifting ftraight lines FE,
BL, GA, KH ; XN, DO, MC, RP not be at right angles to
the bafes of the folids ; and from the points F, B, K, G ; X,
D, R, M draw perpendiculars to the planes in which are the
bafes EH, NP meeting thofe planes in the points S, Y, V, T ;
Q, I, U, Z ; and complete the folids FV, XU, which are pa-
rallelepipeds, as was proved in the laft part of prop. 31. of
this book. In this cafe, likewife, if the folids AB, CD be e-
qual, their bafes are reciprocally proportional to their altitudes,
viz. the bafe EH to the bafe NP, as the altitude of the folid
CD to the altitude of the folid AB. Becaufe the folid AB is
equal to the folid CD, and that the folid BT is equal ᵍ to the g 29. or 30.
folid BA, for they are upon the fame bafe FK, and of the ¹¹·
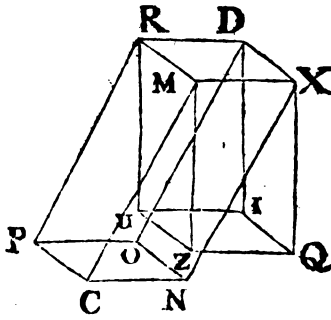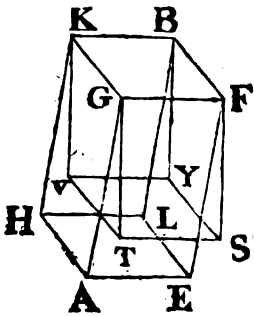
fame altitude ; and that the folid DC is equal ᵍ to the folid
DZ, being upon the fame bafe XR, and of the fame altitude ;
therefore the folid BT is equal to the folid DZ : But the bafes
are reciprocally proportional to the altitudes of equal folid pa-
rallelepipeds of which the infifting ftraight lines are at right
angles to their bafes, as before was proved : Therefore as the
bafe FK to the bafe XR, fo is the altitude of the folid DZ to
the altitude of the folid BT : And the bafe FK is equal to the
bafe EH, and the bafe XR to the bafe NP : Wherefore, as the
bafe EH to the bafe NP, fo is the altitude of the folid DZ to
the altitude of the folid BT : But the altitudes of the folids
DZ, DC, as alfo of the folids BT, BA are the fame. There-
fore, as the bafe EH to the bafe NP, fo is the altitude of the

Q   folid

folid CD to the altitude of the folid AB ; that is, the bafes of the folid parallelepipeds AB, CD are reciprocally proportional to their altitudes.

Next, Let the bafes of the folids AB, CD be reciprocally proportional to their altitudes, viz. the bafe EH to the bafe NP, as the altitude of the folid CD to the altitude of the folid AB ; the folid AB is equal to the folid CD : The fame conftruction being made ; becaufe, as the bafe EH to the bafe NP, fo is the altitude of the folid CD to the altitude of the folid AB ; and that the bafe EH is equal to the bafe FK ; and NP to XR ; therefore the bafe FK is to the bafe XR, as the altitude of the folid CD to the altitude of AB : But the alti-

tudes of the folids AB, BT are the fame, as alfo of CD and DZ ; therefore, as the bafe FK to the bafe XR, fo is the altitude of the folid DZ to the altitude of the folid BT : Wherefore the bafes of the folids BT, DZ are reciprocally proportional to their altitudes ; and their infifting ftraight lines are at right angles to the bafes ; wherefore, as was before proved, the folid BT is equal to the folid DZ : But BT is equal [g] to the folid BA, and DZ to the folid DC, becaufe they are upon the fame bafes, and of the fame altitude. Therefore the folid AB is equal to the folid CD. Q. E. D.
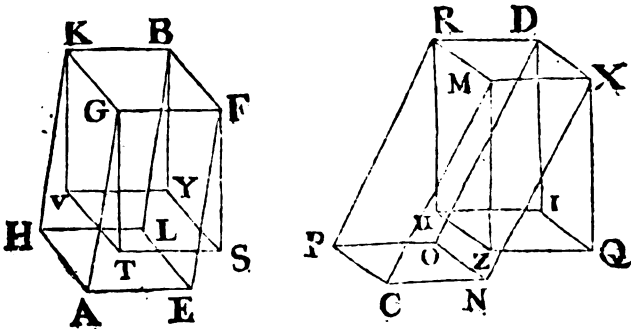
g 29. or 30. 11.

P R O P.

PROP. XXXV. THEOR.

IF, from the vertices of two equal plane angles, there
be drawn two ſtraight lines elevated above the planes
in which the angles are, and containing equal angles with
the ſides of thoſe angles, each to each ; and if in the
lines above the planes there be taken any points, and
from them perpendiculars be drawn to the planes in
which the firſt named angles are :   And from the points
in which they meet the planes, ſtraight lines be drawn to
the vertices of the angles firſt named ; theſe ſtraight
lines ſhall contain equal angles with the ſtraight lines
which are above the planes of the angles.

Let BAC, EDF be two equal plane angles ; and from the
points A, D let the ſtraight lines AG, DM be elevated above
the planes of the angles, making equal angles with their ſides
each to each, viz. the angle GAB equal to the angle MDE, and
GAC to MDF ; and in AG, DM let any points G, M be ta-
ken, and from them let perpendiculars GL, MN be drawn to
the planes BAC, EDF meeting theſe planes in the points L, N ;
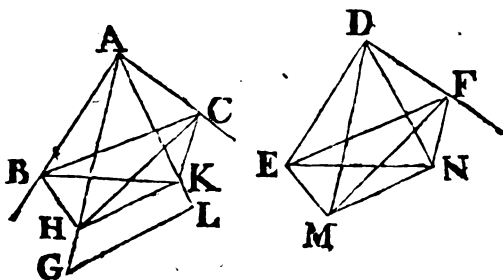
and join LA, ND : The angle GAL is equal to the angle
MDN.
Make AH equal to DM, and through H draw HK parallel
to GL : But GL is perpendicular to the plane BAC ; where-
fore HK is perpendicular [a] to the ſame plane : From the points
K, N, to the ſtraight lines AB, AC, DE, DF, draw perpen-
diculars KB, KC, NE, NF ; and join HB, BC, ME, EF :

Q 2   · Becauſe

Becaufe HK is perpendicular to the plane BAC, the plane
HBK which paſſes through HK is at right angles [b] to the plane
BAC ; and AB is drawn in the plane BAC at right angles to
the common fection BK of the two planes ; therefore AB is
perpendicular [c] to the plane HBK, and makes right angles [d]
with every ſtraight line meeting it in that plane : But BH meets
it in that plane ; therefore ABH is a right angle : For the fame
reafon, DEM is a right angle, and is therefore equal to the
angle ABH : And the angle HAB is equal to the angle MDE.
Therefore in the two triangles HAB, MDE there are two angles
in one equal to two angles in the other, each to each, and
one fide equal to one fide, oppofite to one of the equal angles
in each, viz. HA equal to DM ; therefore the remaining fides
are equal [e], each to each :   Wherefore AB is equal to DE.   In
the fame manner, if HC and MF be joined, it may be demon-
ftrated that AC is equal to DF : Therefore, fince AB is equal to
DE, BA and AC are equal to ED and DF ; and the angle
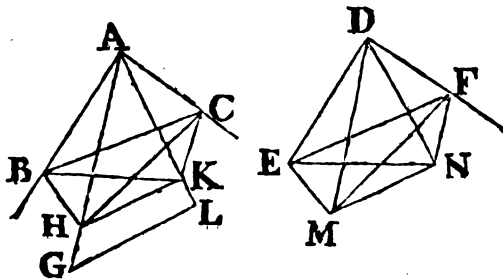
BAC is equal to the angle EDF ; wherefore the bafe BC is e-
qual [f] to the bafe EF, and the remaining angles to the remain-
ing angles :   The angle ABC is therefore equal to the angle
DEF :   And the right angle ABK is equal to the right angle
DEN, whence the remaining angle CBK is equal to the re-
maining angle FEN :   For the fame reafon, the angle BCK is
equal to the angle EFN : Therefore, in the two triangles BCK,
EFN, there are two angles in one equal to two angles in the
other, each to each, and one fide equal to one fide adjacent
to the equal angles in each, viz. BC equal to EF ; the other
fides, therefore, are equal to the other fides ; BK then is equal
to EN : And AB is equal to DE ; wherefore AB, BK are equal
to DE, EN ; and they contain right angles ; wherefore the
bafe AK is equal to the bafe DN :   And fince AH is equal to
                                                        DM,

DM, the fquare of AH is equal to the fquare of DM: But the
fquares of AK, KH are equal to the fquare ^{s} of AH, becaufe
AKH is a right angle: And the fquares of DN, NM are equal ^{s} 47. 1.
to the fquare of DM, for DNM is a right angle: Wherefore
the fquares of AK, KH are equal to the fquares of DN, NM;
and of thofe the fquare of AK is equal to the fquare of DN:
Therefore the remaining fquare of KH is equal to the remain-
ing fquare of NM; and the ftraight line KH to the ftraight
line NM: And becaufe HA, AK are equal to MD, DN, each
to each, and the bafe HK to the bafe MN, as has been proved;
therefore the angle HAK is equal ^{h} to the angle MDN. Q. ^{h} 8. 1.
E. D.

Cor. From this it is manifeft, that if, from the vertices of
two equal plane angles, there be elevated two equal ftraight lines
containing equal angles with the fides of the angles, each to
each; the perpendiculars drawn from the extremities of the e-
qual ftraight lines to the planes of the firft angles are equal to
one another.

Another Demonftration of the Corollary.

Let the plane angles BAC, EDF be equal to one another,
and let AH, DM be two equal ftraight lines above the planes of
the angles, containing equal angles with BA, AC; ED, DF,
each to each, viz. the angle HAB equal to MDE, and HAC
equal to the angle MDF; and from H, M let HK, MN be per-
pendiculars to the planes BAC, EDF: HK is equal to MN.

Becaufe the folid angle at A is contained by the three plane
angles BAC, BAH, HAC, which are, each to each, equal to
the three plane angles EDF, EDM, MDF containing the folid
angle at D; the folid angles at A and D are equal; And there-
fore coincide with one another; to wit, if the plane angle BAC
be applied to the plane angle EDF, the ftraight line AH coin-
cides with DM, as was fhewn in prop. B of this book: And
becaufe AH is equal to DM, the point H coincides with the
point M: Wherefore HK which is perpendicular to the plane
BAC coincides with ^{i} MN which is perpendicular to the plane ^{i} 13. 11.
EDF, becaufe thefe planes coincide with one another: There-
fore HK is equal to MN. Q. E. D.

Q 3   P R O P.

PROP. XXXVI. THEOR.

See N.

IF three ftraight lines be proportionals, the folid paral-
lelepiped defcribed from all three as its fides, is equal
to the equilateral parallelepiped defcribed from the mean
proportional, one of the folid angles of which is contain-
ed by three plane angles equal, each to each, to the three
plane angles containing one of the folid angles of the o-
ther figure.

Let A, B, C be three proportionals, viz. A to B, as B to
C. The folid defcribed from A, B, C is equal to the equila-
teral folid defcribed from B, equiangular to the other.

Take a folid angle D contained by three plane angles EDF,
FDG, GDE; and make each of the ftraight lines ED, DF,
DG equal to B, and complete the folid parallelepiped DH:
Make LK equal to A, and at the point K in the ftraight line
a 26. 11. LK make a folid angle contained by the three plane angles
LKM, MKN, NKL equal to the angles EDF, FDG, GDE,
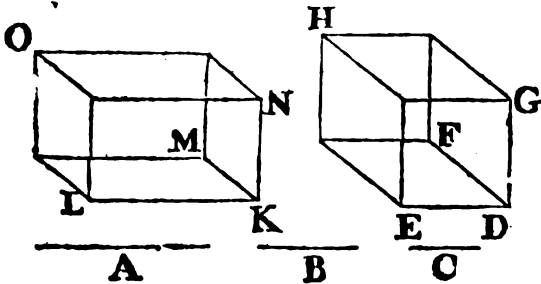
each to each; and make KN equal to B, and KM, equal to
C; and complete the folid parallelepiped KO: And becaufe, as
A is to B, fo is B to C, and that A is equal to LK, and B
to each of the ftraight lines DE, DF, and C to KM; there-
fore LK is to ED, as DF to KM; that is, the fides about the
equal angles are reciprocally proportional; therefore the pa-
b 14. 6. rallelogram LM is equal b to EF: And becaufe EDF, LKM are
two equal plane angles, and the two equal ftraight lines DG,
KN are drawn from their vertices above their planes, and con-
tain equal angles with their fides; therefore the perpendicu-
lars from the points G, N, to the planes EDF, LKM are e-
qual

qual[c] to one another: Therefore the folids KO, DH are of
the fame altitude; and they are upon equal bafes LM, EF,
and therefore they are equal[d] to one another: But the folid
KO is defcribed from the three ftraight lines A, B, C, and the
folid DH from the ftraight line B. If therefore three ftraight
lines, &c. Q. E. D.

P R O P. XXXVII.   T H E O R.

See N.

IF four ftraight lines be proportionals, the fimilar folid
parallelepipeds fimilarly defcribed from them fhall alfo
be proportionals. And if the fimilar parallelepipeds fimi-
larly defcribed from four ftraight lines be proportionals,
the ftraight lines fhall be proportionals.

Let the four ftraight lines AB, CD, EF, GH be proporti-
onals, viz. as AB to CD, fo EF to GH; and let the fimilar
parallelepipeds AK, CL, EM, GN be fimilarly defcribed from
them. AK is to CL, as EM to GN.

Make [a] AB, CD, O, P continual proportionals, as alfo EF,
GH, Q, R: And becaufe as AB is to CD, fo EF to GH; and

a 11. 6.
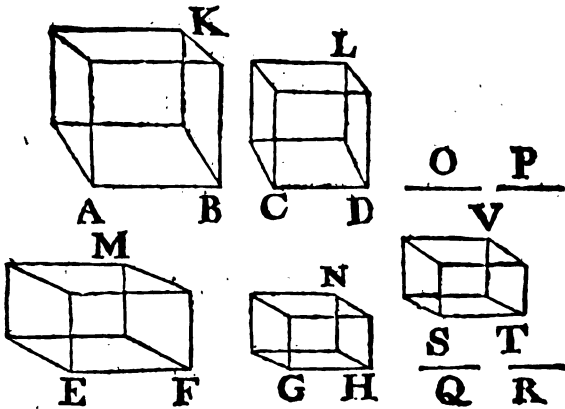
that CD is [b] to O, as GH to Q, and O to P, as Q to R; there-
fore, ex aequali[c], AB is to P, as EF to R: But as AB to P,
fo [d] is the folid AK to the folid CL; and as EF to R, to[d] is
the folid EM to the folid GN: Therefore[b] as the folid AK to
the folid CL, fo is the folid EM to the folid GN.

b 11. 5.
c 22. 5.
d Cor. 33.
11.

Q 4   But

But let the folid AK be to the folid CL, as the folid EM to the folid GN: The ftraight line AB is to CD, as EF to GH.

Take AB to CD, as EF to ST, and from ST defcribe [e] a folid parallelepiped SV fimilar and fimilarly fituated to either of the folids EM, GN: And becaufe AB is to CD, as EF to ST, and that from AB, CD the folid parallelepipeds AK, CL are fimilarly defcribed; and in like manner the folids EM, SV from the ftraight lines EF, ST; therefore AK is to CL, as
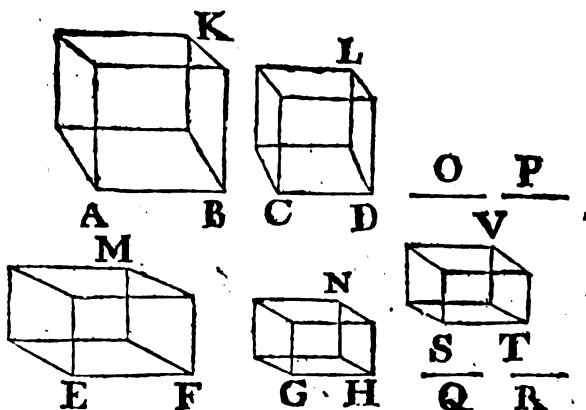
EM to SV: But, by the hypothefis, AK is to CL, as EM to GN: Therefore GN is equal [f] to SV: But it is likewife fimilar and fimilarly fituated to SV; therefore the planes which contain the folids GN, SV are fimilar and equal, and their homologous fides GH, ST equal to one another: And becaufe as AB to CD, fo EF to ST, and that ST is equal to GH; AB is to CD, as EF to GH. Therefore, if four ftraight lines, &c. Q. E. D.

PROP. XXXVIII. THEOR.

" IF a plane be perpendicular to another plane, and
" a ftraight line be drawn from a point in one of
" the planes perpendicular to the other plane, this ftraight
" line fhall fall on the common fection of the planes.

" Let the plane CD be perpendicular to the plane AB, and
" let AD be their common fection; if any point E be taken in
" the plane CD, the perpendicular drawn from E to the plane
" AB fhall fall on AD.

" For,

" For, if it does not, let it, if possible, fall elsewhere, as EF; **Book XI.**
" and let it meet the plane AB in the point F; and from F
" draw ª, in the plane AB, a perpendicular FG to DA, which a 12. 1.
" is also perpendicular ᵇ to the plane CD; and join EG: Then b4. def. 11.
" because FG is perpendicular
" to the plane CD, and the
" straight line EG, which is in
" that plane, meets it; there-   c 3. def. 11.
" fore FGE is a right angle ᶜ:
" But EF is also at right angles
" to the plane AB; and there-
" fore EFG is a right angle:
" Wherefore two of the angles
" of the triangle EFG are equal together to two right angles;
" which is absurd: Therefore the perpendicular from the point
" E to the plane AB does not fall elsewhere than upon the
" straight line AD: It therefore falls upon it. If therefore a
" plane," &c. Q. E. D.
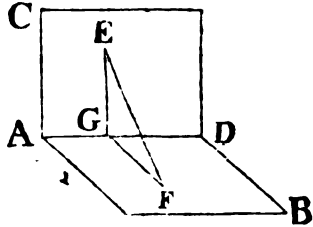
PROP. XXXIX. THEOR.

IN a solid parallelepiped, if the sides of two of the op- See N.
posite planes be divided each into two equal parts, the
common section of the planes passing through the points
of division, and the diameter of the solid parallelepiped
cut each other into two equal parts.

Let the sides of
the opposite planes
CF, AH of the so-
lid parallelepiped
AF, be divided
each into two e-
qual parts in the
points K, L, M,
N; X, O, P, R:
and join KL, MN,
XO, PR: And be-
cause DK, CL are
equal and parallel,
KL is parallel ª to
DC: For the same
reason, MN is pa-
rallel to BA: And   a 33. 1.

9. 11.
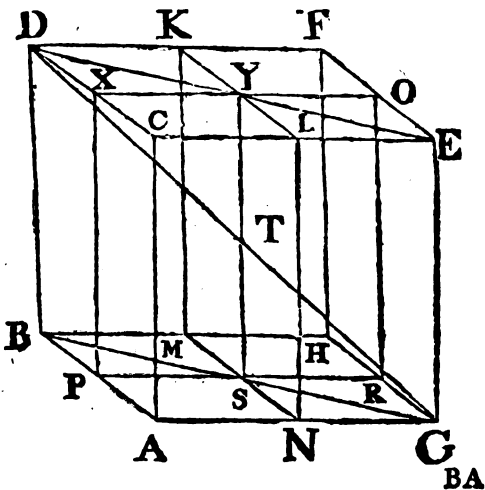
BA is parallel to DC; therefore, becaufe KL, BA are each of them parallel to DC, and not in the fame plane with it, KL is parallel [b] to BA: And becaufe KL, MN are each of them parallel to BA, and not in the fame plane with it, KL is parallel [b] to MN; wherefore KL, MN are in one plane. In like manner, it may be proved, that XO, PR are in one plane. Let YS be the common fection of the planes KN, XR; and DG the diameter of the folid parallelepiped AF: YS and DG do meet, and cut one another into two equal parts.

Join DY, YE, BS, SG. Becaufe DX is parallel to OE, the

c 29. 1.

alternate angles DXY, YOE are equal [c] to one another: And becaufe DX is e-qual to OE, and XY to YO, and contain equal angles, the bafe DY

d 4. 1.

is equal [d] to the bafe YE, and the other angles are equal; therefore the angle XYD is equal to the angle OYE, and DYE is a ftraight

e 14. 1.

[e] line: For the fame reafon BSG is a ftraight line, and BS equal to
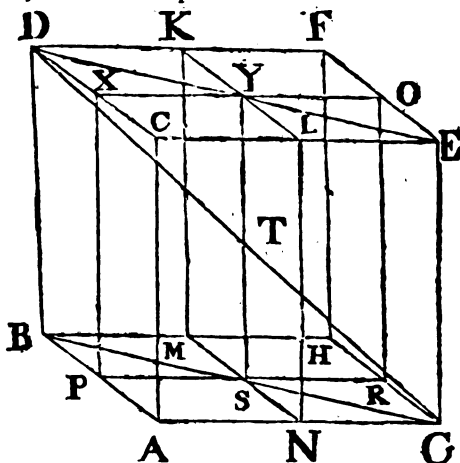
SG: And becaufe CA is equal and parallel to DB, and alfo equal and parallel to EG; therefore DB is equal and parallel [b] to EG: And DE, BG join their extremities; therefore DE is

a 33. 1.

equal and parallel [a] to BG: And DG, YS are drawn from points in the one, to points in the other; and are therefore in one plane: Whence it is manifeft, that DG, YS muft meet one another; let them meet in T: And becaufe DE is parallel to BG, the alternate angles EDT, BGT are equal [e]; and the

f 15. 1.

angle DTY is equal [f] to the angle GTS: Therefore in the triangles DTY, GTS there are two angles in the one equal to two angles in the other, and one fide equal to one fide, oppofite to two of the equal angles, viz. DY to GS; for they are the

g 26. 1.

halves of DE, BG: Therefore the remaining fides are equal [g], each to each. Wherefore DT is equal to TG, and YT equal to TS. Therefore, if in a folid, &c. Q. E. D.

PROP. XL. THEOR.

IF there be two triangular prifms of the fame altitude, the bafe of one of which is a parallelogram, and the bafe of the other a triangle; if the parallelogram be double of the triangle, the prifms fhall be equal to one another.

Let the prifms ABCDEF, GHKLMN be of the fame altitude, the firſt whereof is contained by the two triangles ABE, CDF, and the three parallelograms AD, DE, EC; and the other by the two triangles GHK, LMN and the three parallelograms LH, HN, NG; and let one of them have a parallelogram AF, and the other a triangle GHK for its bafe; if the parallelogram AF be double of the triangle GHK, the prifm ABCDEF is equal to the prifm GHKLMN.

Complete the folids AX, GO; and becaufe the parallelogram AF is double of the triangle GHK; and the parallelo-

gram HK double [a] of the fame triangle; therefore the paral-   a 34. 1.
lelogram AF is equal to HK. But folid parallelepipeds upon
equal bafes, and of the fame altitude, are equal [b] to one an-   b 31. 11.
other. Therefore the folid AX is equal to the folid GO; and
the prifm ABCDEF is half [c] of the folid AX; and the prifm   c 28. 11.
GHKLMN half [c] of the folid GO. Therefore the prifm
ABCDEF is equal to the prifm GHKLMN. Wherefore, if
there be two, &c. Q. E. D.
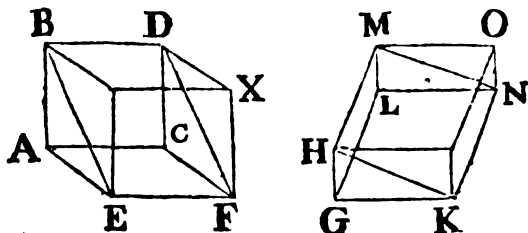
THE

ELEMENTS

OF

EUCLID.

BOOK XII.

LEMMA I.

Which is the firſt propoſition of the tenth book, and is neceſ-
ſary to ſome of the propoſitions of this book.

IF from the greater of two unequal magnitudes, there
be taken more than its half, and from the remainder
more than its half; and ſo on: There ſhall at length re-
main a magnitude leſs than the leaſt of the propoſed
magnitudes.

Let AB and C be two unequal magnitudes, of which AB is
the greater. If from AB there be taken more
than its half, and from the remainder more than
its half, and ſo on; there ſhall at length remain
a magnitude leſs than C.

For C may be multiplied ſo, as at length to
become greater than AB. Let it be ſo multi-
plied, and let DE its multiple be greater than
AB, and let DE be divided into DF, FG, GE,
each equal to C. From AB take BH greater
than its half, and from the remainder AH
take HK greater than its half, and ſo on, until
there be as many diviſions in AB as there are
in DE: And let the diviſions in AB be AK,
KH, HB; and the diviſions in DE be DF, FG,
GE. And becauſe DE is greater than AB, and
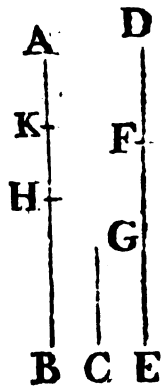
that

that EG taken from DE is not greater than its half, but BH ta-
ken from AB is greater than its half ; therefore the remainder
GD is greater than the remainder HA. , Again, becaufe GD is
greater than HA, and that GF is not greater than the half of
GD, but HK is greater than the half of HA ; therefore the re-
mainder FD is greater than the remainder AK.   And FD is
equal to C, therefore C is greater than AK; that is, AK is lefs
than C.   Q. E. D.
    And if only the halves be taken away, the fame thing may in
the fame way be demonftrated.

PROP. I.   THEOR.

SIMILAR polygons infcribed in circles, are to one an-
other as the fquares of their diameters.

    Let ABCDE, FGHKL be two circles, and in them the fi-
milar polygons ABCDE, FGHKL ; and let BM, GN be the
diameters of the circles : As the fquare of BM is to the fquare
of GN, fo is the polygon ABCDE to the polygon FGHKL.
    Join BE, AM, GL, FN : And becaufe the polygon ABCDE is
fimilar to the polygon FGHKL, and fimilar polygons are divided
into fimilar triangles ; the triangles ABE, FGL are fimilar and e-

quiangular [b] ; and therefore the angle AEB is equal to the angle [b 6. 7.]
FLG : But AEB is equal [c] to AMB, becaufe they ftand up- [c 21. 3.]
on the fame circumference ; and the angle FLG is, for the fame
reafon, equal to the angle FNG : Therefore alfo the angle AMB
is equal to FNG : And the right angle BAM is equal to the
right [d] angle GFN ; wherefore the remaining angles in the tri- [d 31. 3.]
angles ABM, FGN are equal, and they are equiangular to one
another ;
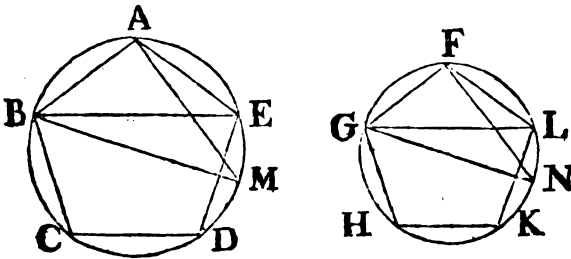
e 4. 6.
f 10. def.
5. & 22.
5.
g 20. 6.

another: Therefore as BM to GN, fo ᵉ is BA to GF; and there-
fore the duplicate ratio of BM to GN, is the fame ᶠ with the du-
plicate ratio of BA to GF: But the ratio of the fquare of BM to
the fquare of GN, is the duplicate ᵍ ratio of that which BM has
to GN; and the ratio of the polygon ABCDE to the polygon
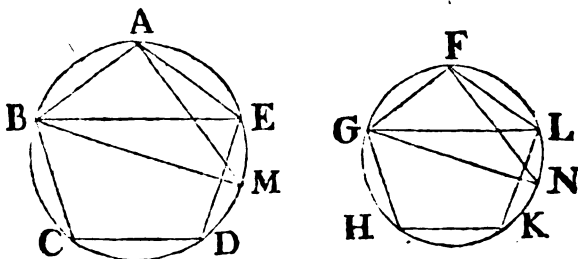
FGHKL is the duplicate ᵍ of that which BA has to GF: There-
fore as the fquare of BM to the fquare of GN, fo is the polygon
ABCDE to the polygon FGHKL. Wherefore fimilar polygons,
&c. Q. E. D.

PROP. II. THEOR.

See N.

CIRCLES are to one another as the fquares of their
diameters.

Let ABCD, EFGH be two circles, and BD, FH their dia-
meters: As the fquare of BD to the fquare of FH, fo is the
circle ABCD to the circle EFGH.

For, if it be not fo, the fquare of BD fhall be to the fquare
of FH, as the circle ABCD is to fome fpace either lefs than the
circle EFGH, or greater than it *. Firft, let it be to a fpace
S lefs than the circle EFGH; and in the circle EFGH
defcribe the fquare EFGH: This fquare is greater than
half of the circle EFGH; becaufe if, through the points
E, F, G, H, there be drawn tangents to the circle, the
fquare

---

* For there is fome fquare equal to
the circle ABCD; let P be the fide of
it, and to three ftraight lines BD, FH
and P, there can be a fourth propor-
tional; let this be Q: Therefore the
fquares of thefe four ftraight lines are
proportionals; that is, to the fquares of
BD, FH and the circle ABCD, it is pof-
fible there may be a fourth proportiog-
al. Let this be S. And in like man-
ner are to be underftood fome things
in fome of the following propofitions.

square EFGH is half [a] of the square defcribed about the circle ;
and the circle is lefs than the fquare defcribed about it ; there-
fore the fquare EFGH is greater than half of the circle. Di- [a 47. 1.]
vide the circumferences EF, FG, GH, HE, each into two equal
parts in the points K, L, M, N, and join EK, KF, FL, LG, GM,
MH, HN, NE: Therefore each of the triangles EKF, FLG,
GMH, HNE is greater than half of the fegment of the circle
it ftands in ; becaufe, if ftraight lines touching the circle be
drawn through the points K, L. M, N, and parallelograms up-
on the ftraight lines EF, FG, GH, HE be completed ; each
of the triangles EKF, FLG, GMH, HNE fhall be the half [a 41. 1.]
of the parallelogram in which it is : But every fegment is lefs
than the parallelogram in which it is : Wherefore each of the
triangles EKF, FLG, GMH, HNE is greater than half the
fegment of the circle which contains it : And if thefe cir-
cumferences before named be divided each into two equal parts,
and their extremities be joined by ftraight lines, by continuing
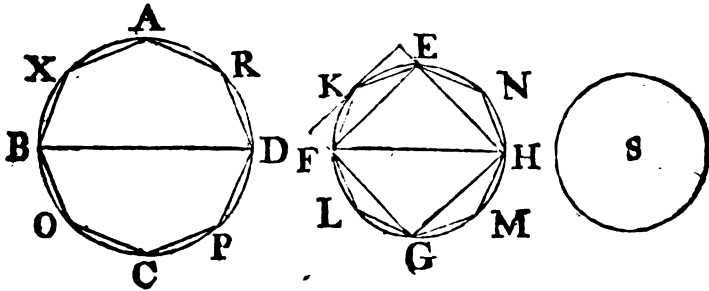
to do this, there will at length remain fegments of the circle
which, together, fhall be lefs than the excefs of the circle EFGH
above the fpace S : Becaufe, by the preceding lemma, if
from the greater of two unequal magnitudes there be taken
more than its half, and from the remainder more than its
half, and fo on, there fhall at length remain a magnitude lefs
than the leaft of the propofed magnitudes. Let then the feg-
ments EK, KF, FL, LG, GM, MH, HN, NE be thofe that
remain and are together lefs than the excefs of the circle EFGH
above S : Therefore the reft of the circle, viz. the polygon
EKFLGMHN, is greater than the fpace S. Defcribe likewife
in the circle ABCD the polygon AXBOCPDR fimilar to the
polygon EKFLGMHN : As, therefore, the fquare of BD is to
the fquare of FH, fo [h] is the polygon AXBOCPDR to the po- [b 1. 12]
lygon EKFLGMHN : But the fquare of BD is alfo to the
                                                    fquare

square of FH, as the circle ABCD is to the space S: There-
fore as the circle ABCD is to the space S, fo is ᶜ the polygon
AXBOCPDR to the polygon EKFLGMHN: But the circle
ABCD is greater than the polygon contained in it; wherefore
the space S is greater ᵈ than the polygon EKFLGMHN: But
it is likewife lefs, as has been demonftrated; which is impoffi-
ble. Therefore the square of BD is not to the square of FH,
as the circle ABCD is to any fpace lefs than the circle EFGH.
In the fame manner, it may be demonftrated, that neither is the
fquare of FH to the fquare of BD, as the circle EFGH is to
any fpace lefs than the circle ABCD. Nor is the fquare of
BD to the fquare of FH, as the circle ABCD is to any fpace
greater than the circle EFGH: For, if poffible, let it be fo to
T, a fpace greater than the circle EFGH: Therefore, inverfely,
as the fquare of FH to the fquare of BD, fo is the fpace T to
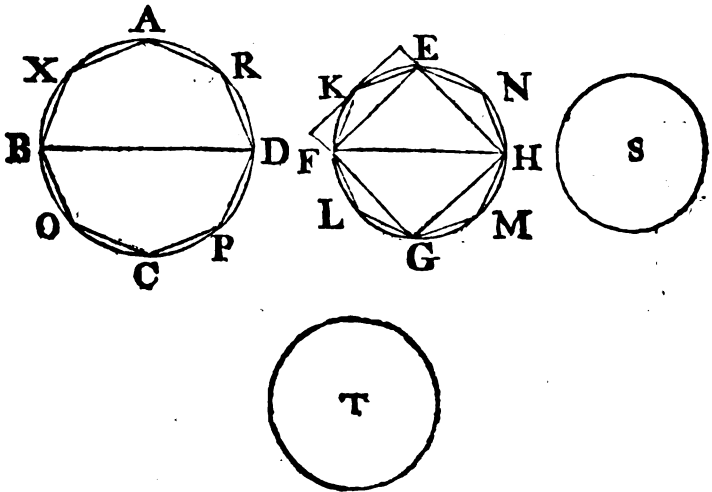
the circle ABCD.   But as the fpace † T is to the circle ABCD,
fo is the circle EFGH to fome fpace, which muft be lefs ᵈ than
the circle ABCD, becaufe the fpace T is greater, by hypothe-
fis, than the circle EFGH.   Therefore as the fquare of FH is to
the

† For as, in the foregoing note, at °,
it was explained how it was poffible
there could be a fourth proportional to
the fquares of BD, FH, and the circle
ABCD, which was named S. So, in like
manner, there can be a fourth propor-
tional to this other fpace, named T,
and the circles ABCD, EFGH. And
the like is to be underftood in fome of
the following propofitions.

the fquare of BD, fo is the circle EFGH to a fpace lefs than the circle ABCD, which has been demonftrated to be impof-fible : Therefore the fquare of BD is not to the fquare of FH, as the circle ABCD is to any fpace greater than the circle EFGH: And it has been demonftrated, that neither is the fquare of BD to the fquare of FH, as the circle ABCD to any fpace lefs than the circle EFGH : Wherefore, as the fquare of BD to the fquare of FH, fo is the circle ABCD to the circle EFGH ‡. Circles therefore are, &c. Q. E. D.

PROP. III. THEOR.

EVERY pyramid having a triangular bafe, may be divided into two equal and fimilar pyramids having triangular bafes, and which are fimilar to the whole pyramid ; and into two equal prifms which together are greater than half of the whole pyramid.

Let there be a pyramid of which the bafe is the triangle ABC and its vertex the point D : The pyramid ABCD may be divided into two equal and fimilar pyramids having triangular bafes, and fimilar to the whole ; and into two equal prifms which together are greater than half of the whole pyramid.
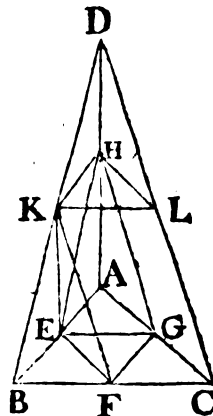
Divide AB, BC, CA, AD, DB, DC, each into two equal parts in the points E, F, G, H, K, L, and join EH, EG, GH, HK, KL, LH, EK, KF, FG. Becaufe AE is equal to EB, and AH to HD, HE is parallel [a] to DB : For the fame reafon, HK is parallel to AB : Therefore HEBK is a parallelogram, and HK equal [b] to EB : But EB is equal to AE ; therefore alfo AE is equal to HK: And AH is equal to HD; wherefore EA, AH are equal to KH, HD, each to each ; and the angle EAH is equal [c] to the angle KHD; therefore the bafe EH is equal to the bafe KD, and the triangle

a 2. 6.

b 34. 1.

c 29. 1.

R   AEH

‡ Becaufe as a fourth proportional to the fquares of BD, FH and the circle ABCD is poffible, and that it can neither be lefs nor greater than the circle EFGH, it muft be equal to it.

AEH equal ^d and fimilar to the triangle HKD : For the fame reafon, the triangle AGH is equal and fimilar to the triangle HLD : And becaufe the two ftraight lines EH, HG which meet one another are parallel to KD, DL that meet one another, and are not in the fame plane with them, they contain

equal ^e angles ; therefore the angle EHG is equal to the angle KDL. Again, becaufe EH, HG are equal to KD, DL, each to each, and the angle EHG equal to the angle KDL ; therefore the bafe EG is equal to the bafe KL : And the triangle EHG equal ^d and fimilar to the triangle KDL : For the fame reafon, the triangle AEG is alfo equal and fimilar to the triangle HKL. Therefore the pyramid of which the bafe is the triangle AEG, and of which the vertex is the point H, is e-

qual ^f and fimilar to the pyramid the bafe of which is the triangle KHL, and vertex the point D : And becaufe HK is parallel to AB a fide of the triangle ADB, the triangle ADB is equiangular to the triangle HDK, and their

fides are proportionals ^g : Therefore the triangle ADB is fimilar to the triangle HDK : And for the fame reafon, the triangle DBC is fimilar to the triangle DKL ; and the triangle ADC to the triangle HDL ; and alfo the triangle ABC to the triangle AEG : But the triangle AEG is fimilar to the triangle HKL, as before was proved ; therefore

the triangle ABC is fimilar ^h to the triangle HKL. And the pyramid of which the bafe is the triangle ABC, and vertex the point D,

is therefore fimilar ⁱ to the pyramid of which the bafe is the triangle HKL, and vertex the fame point D : But the pyramid of which the bafe is the triangle HKL, and vertex the point D, is fimilar, as has been proved, to the pyramid the bafe of which is the triangle AEG, and vertex the point H : Wherefore the pyramid the bafe of which is the triangle ABC, and vertex the point D, is fimilar to the pyramid of which the bafe is the triangle AEG and vertex H : Therefore each of the pyramids AEGH, HKLD is fimilar to the whole pyramid ABCD : And becaufe

BF is equal to FC, the parallelogram EBFG is double ^k of the triangle GFC : But when there are two prifms of the fame altitude,
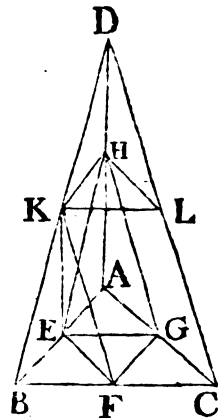
tude, of which one has a parallelogram for its base, and the other
a triangle that is half of the parallelogram, these prisms are equal
^a to one another; therefore the prism having the parallelogram
EBFG for its base, and the straight line KH opposite to
it, is equal to the prism having the triangle GFC for its base,
and the triangle HKL opposite to it; for they are of the same
altitude, because they are between the parallel ^b planes ABC,
HKL: And it is manifest that each of these prisms is greater
than either of the pyramids of which the triangles AEG, HKL
are the bases, and the vertices the points H, D; because, if
EF be joined, the prism having the parallelogram EBFG for
its base, and KH the straight line opposite to it, is greater than
the pyramid of which the base is the triangle EBF, and vertex
the point K; but this pyramid is equal ^c to the pyramid the
base of which is the triangle AEG, and vertex the point H;
because they are contained by equal and similar planes: Where-
fore the prism having the parallelogram EBFG for its base,
and opposite side KH, is greater than the pyramid of which
the base is the triangle AEG, and vertex the point H: And the
prism of which the base is the parallelogram EBFG, and op-
posite side KH is equal to the prism having the triangle GFC
for its base, and HKL the triangle opposite to it; and the pyra-
mid of which the base is the triangle AEG, and vertex H, is
equal to the pyramid of which the base is the triangle HKL,
and vertex D: Therefore the two prisms before mentioned are
greater than the two pyramids of which the bases are the tri-
angles AEG, HKL, and vertices the points H, D. Therefore
the whole pyramid of which the base is the triangle ABC, and
vertex the point D, is divided into two equal pyramids similar
to one another, and to the whole pyramid; and into two equal
prisms; and the two prisms are together greater than half of the
whole pyramid.   Q. E. D.

PROP. IV. THEOR.

See N.

IF there be two pyramids of the fame altitude, upon triangular bafes, and each of them be divided into two equal pyramids fimilar to the whole pyramid, and alfo into two equal prifms ; and if each of thefe pyramids be divided in the fame manner as the firft two, and fo on : As the bafe of one of the firft two pyramids is to the bafe of the other, fo fhall all the prifms in one of them be to all the prifms in the other, that are produced by the fame number of divifions.

Let there be two pyramids of the fame altitude upon the tri-angular bafes ABC, DEF, and having their vertices in the points G, H ; and let each of them be divided into two equal pyramids fimilar to the whole, and into two equal prifms ; and let each of the pyramids thus made be conceived to be divided in the like manner, and fo on : As the bafe ABC is to the bafe DEF, fo are all the prifms in the pyramid ABCG to all the prifms in the pyramid DEFH made by the fame number of di-vifions.

Make the fame conftruction as in the foregoing propofition : and becaufe BX is equal to XC, and AL to LC, therefore XL

a 2. 6.

is parallel ᵃ to AB, and the triangle ABC fimilar to the tri-angle LXC : For the fame reafon, the triangle DEF is fimilar to RVF : And becaufe BC is double of CX, and EF double of FV, therefore BC is to CX, as EF to FV : And upon BC, CX are defcribed the fimilar and fimilarly fituated rectilineal fi-gures ABC, LXC ; and upon EF, FV, in like manner, are defcribed the fimilar figures DEF, RVF : Therefore, as the tri-

b 22. 6.

angle ABC is to the triangle LXC, fo ᵇ is the triangle DEF to the triangle RVF, and, by permutation, as the triangle ABC to the triangle DEF, fo is the triangle LXC to the triangle RVF : And becaufe the planes ABC, OMN, as alfo the planes

c 15. 11.

DEF, STY are parallel ᶜ, the perpendiculars drawn from the points G, H to the bafes ABC, DEF, which, by the hypothe-fis, are equal to one another, fhall be cut each into two equal

d 17. 11.

ᵈ parts by the planes OMN, STY, becaufe the ftraight lines GC, HF are cut into two equal parts in the points N, Y by the fame planes : Therefore the prifms LXCOMN, RVFSTY are of the fame altitude ; and therefore, as the bafe LXC to

the

the bafe RVF; that is, as the triangle ABC to the triangle DEF, fo ᵃ is the prifm having the triangle LXC for its bafe, and OMN the triangle oppofite to it, to the prifm of which the bafe is the triangle RVF, and the oppofite triangle STY: And becaufe the two prifms in the pyramid ABCG are equal to one another, and alfo the two prifms in the pyramid DEFH equal to one another, as the prifm of which the bafe is the pa- rallelogram KBXL and oppofite fide MO, to the prifm having the triangle LXC for its bafe, and OMN the triangle oppofite to it; fo is the prifm of which the bafe ᵇ is the parallelogram PEVR, and oppofite fide TS, to the prifm of which the bafe is the triangle RVF, and oppofite triangle STY. Therefore, componendo, as the prifms KBXLMO, LXCOMN together
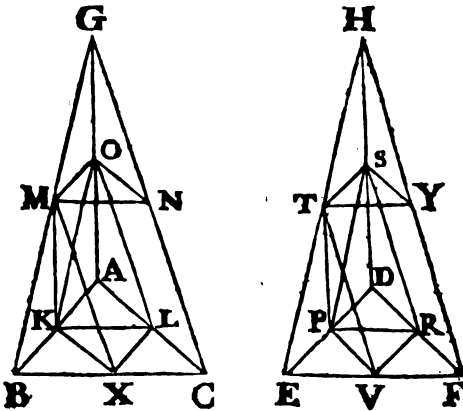
are unto the prifm LXCOMN; fo are the prifms PEVRTS, RVFSTY to the prifm RVFSTY: And, permutando, as the prifms KBXLMO, LXCOMN are to the prifms PEVRTS, RVFSTY; fo is the prifm LXCOMN to the prifm RVFSTY: But as the prifm LXCOMN to the prifm RVFSTY, fo is, as has been proved, the bafe ABC to the bafe DEF: Therefore, as the bafe ABC to the bafe DEF, fo are the two prifms in the pyramid ABCG to the two prifms in the pyramid DEFH: And likewife if the pyramids now made, for example, the two OMNG, STYH be divided in the fame manner; as the bafe OMN is to the bafe STY, fo fhall the two prifms in the py- ramid OMNG be to the two prifms in the pyramid STYH: But the bafe OMN is to the bafe STY, as the bafe ABC to the bafe DEF; therefore, as the bafe ABC to the bafe DEF, fo are

R 3   the

the two prifms in the pyramid ABCG to the two prifms in the pyramid DEFH; and fo are the two prifms in the pyramid OMNG to the two prifms in the pyramid STYH; and fo are all four to all four: And the fame thing may be fhewn of the prifms made by dividing the pyramids AKLO and DPRS, and of all made by the fame number of divifions. Q. E. D.

PROP. V. THEOR.

See N.

PYRAMIDS of the fame altitude which have trian-gular bafes, are to one another as their bafes.

Let the pyramids of which the triangles ABC, DEF are the bafes, and of which the vertices are the points G, H, be of the fame altitude : As the bafe ABC to the bafe DEF, fo is the py-ramid ABCG to the pyramid DEFH.

For, if it be not fo, the bafe ABC muft be to the bafe DEF, as the pyramid ABCG to a folid either lefs than the pyramid DEFH, or greater than it †. Firft, let it be to a folid lefs than it, viz. to the folid Q : And divide the pyramid DEFH into two equal pyramids, fimilar to the whole, and into two equal prifms : Therefore thefe two prifms are greater ᵃ then the half

a 3. 12.

of the whole pyramid. And, again, let the pyramids made by this divifion be in like manner divided, and fo on, until the pyramids which remain undivided in the pyramid DEFH be, all of them together, lefs than the excefs of the pyramid DEFH above the folid Q: Let thefe, for example, be the pyramids DPRS, STYH: Therefore the prifms, which make the reft of the pyramid DEFH, are greater than the folid Q : Divide like-wife the pyramid ABCG in the fame manner, and into as many parts, as the pyramid DEFH : Therefore, as the bafe

b 4 12.

ABC to the bafe DEF, fo ᵇ are the prifms in the pyramid ABCG to the prifms in the pyramid DEFH : But as the bafe ABC to the bafe DEF, fo, by hypothefis, is the pyramid ABCG to the folid Q ; and therefore, as the pyramid ABCG to the folid Q, fo are the prifms in the pyramid ABCG to the prifms in the pyramid DEFH : But the pyramid ABCG is greater

c 14. 5.

than the prifms contained in it; wherefore ᶜ alfo the folid Q is greater than the prifms in the pyramid DEFH. But it is alfo lefs, which is impoffible. Therefore the bafe ABC is not to the

† This may be explained the fame way as at the note * in propofition 2. in the like cafe.

the bafe DEF, as the pyramid ABCG to any folid which is
lefs than the .pyramid DEFH.   In the fame manner it may
be demonftrated, that the bafe DEF is not to the bafe ABC,
as the pyramid DEFH to any folid which is lefs than the py-
ramid ABCG·   Nor can the bafe ABC be to the bafe DEF, as
the pyramid ABCG tô any folid which is greater than the py-
ramid DEFH.   For if it be poffible, let it be fo to a greater,
viz. the folid Z.   And becaufe the bafe ABC is to the bafe DEF
as the pyramid ABCG to the folid Z; by inverfion, as the bafe
DEF to the bafe ABC, fo is the folid Z to the pyramid ABCG.
But as the folid Z is to the pyramid ABCG, fo is the pyramid
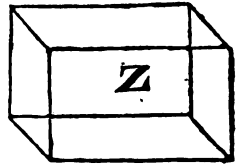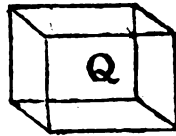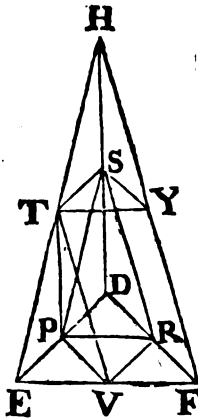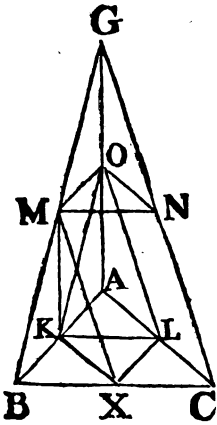
DEFH to fome folid *, which muft be lefs ᵃ than the pyramid   ᵃ 14. 5.
ABCG, becaufe the folid Z is greater than the pyramid DEFH.
And therefore, as the bafe DEF to the bafe ABC, fo is the py-
ramid DEFH to a folid lefs than the pyramid ABCG; the con-
trary to which has been proved.   Therefore the bafe ABC is
not to the bafe DEF, as the pyramid ABCG to any folid which
is greater than the pyramid DEFH.   And it has been proved,
that neither is the bafe ABC to the bafe DEF, as the pyramid
ABCG to any folid which is lefs than the pyramid DEFH.
Therefore, as the bafe ABC is to the bafe DEF, fo is the py-
ramid ABCG to the pyramid DEFH.   Wherefore pyramids,
&c. Q. E. D.

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R 4   P R O P.

</div>

* This may be explained the fame way as the like at the mark † in prop. 2.

PROP. VI. THEOR.

See N. PYRAMIDS of the fame altitude which have polygons for their bafes, are to one another as their bafes.

Let the pyramids which have the polygons ABCDE, FGHKL for their bafes, and their vertices in the points M, N, be of the fame altitude : As the bafe ABCDE to the bafe FGHKL, fo is the pyramid ABCDEM to the pyramid FGHKLN.

Divide the bafe ABCDE into the triangles ABC, ACD, ADE; and the bafe FGHKL into the triangles FGH, FHK, FKL : And upon the bafes ABC, ACD, ADE let there be as many pyramids of which the common vertex is the point M, and upon the remaining bafes as many pyramids having their common vertex in the point N: Therefore, fince the triangle
a 5. 12. ABC is to the triangle FGH, as ª the pyramid ABCM to the pyramid FGHN; and the triangle ACD to the triangle FGH, as the pyramid ACDM to the pyramid FGHN; and alfo the
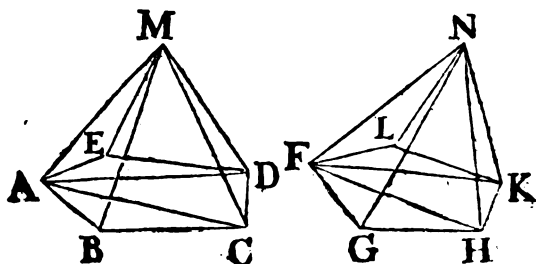
triangle ADE to the triangle FGH, as the pyramid ADEM to the pyramid FGHN; as all the firft antecedents to their com-
b 2. Cor. mon confequent; fo ᵇ are all the other antecedents to their com-
24. 5. mon confequent; that is, as the bafe ABCDE to the bafe FGH, fo is the pyramid ABCDEM to the pyramid FGHN : And, for the fame reafon, as the bafe FGHKL to the bafe FGH, fo is the pyramid FGHKLN to the pyramid FGHN : And, by inverfion, as the bafe FGH to the bafe FGHKL, fo is the pyramid FGHN to the pyramid FGHKLN : Then, becaufe as the bafe ABCDE to the bafe FGH, fo is the pyramid ABCDEM to the pyramid FGHN; and as the bafe FGH to the bafe FGHKL, fo is the pyramid FGHN to the pyramid FGHKLN; therefore,

therefore, ex aequali [e], as the bafe ABCDE to the bafe FGHKL,
fo the pyramid ABCDEM to the pyramid FGHKLN. There-
fore pyramids, &c. Q. E. D.

PROP. VII. THEOR.

EVERY prifm having a triangular bafe may be divi-
ded into three pyramids that have triangular bafes,
and are equal to one another.

Let there be a prifm of which the bafe is the triangle ABC,
and let DEF be the triangle oppofite to it: The prifm ABCDEF
may be divided into three equal pyramids having triangular
bafes.

Join BD, EC, CD; and becaufe ABED is a parallelogram
of which BD is the diameter, the triangle ABD is equal [a] to [a] 34. 1.
the triangle EBD; therefore the pyramid of which the bafe is
the triangle ABD, and vertex the point C, is equal [b] to the [b] 5. 12.
pyramid of which the bafe is the triangle EBD, and vertex the
point C: But this pyramid is the fame with the pyramid the
bafe of which is the triangle EBC, and vertex the point D;
for they are contained by the fame planes: Therefore the py-
ramid of which the bafe is the triangle ABD, and vertex the
point C, is equal to the pyramid, the bafe of which is the tri-
angle EBC, and vertex the point D: Again, becaufe FCBE is
a parallelogram of which the diameter is
CE, the triangle ECF is equal [a] to the
triangle ECB; therefore the pyramid of
which the bafe is the triangle ECB, and
vertex the point D, is equal to the pyra-
mid, the bafe of which is the triangle
ECF, and vertex the point D: But the
pyramid of which the bafe is the triangle
ECB, and vertex the point D has been
proved equal to the pyramid of which the
bafe is the triangle ABD, and vertex the point C. Therefore
the prifm ABCDEF is divided into three equal pyramids having
triangular bafes, viz. into the pyramids ABDC, EBDC, ECFD:
And becaufe the pyramid of which the bafe is the triangle ABD,
and vertex the point C, is the fame with the pyramid of which
the bafe is the triangle ABC, and vertex the point D, for they
are contained by the fame planes; and that the pyramid of which
the bafe is the triangle ABD, and vertex the point C, has been
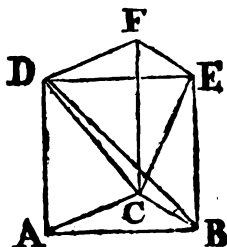
e   demonftrated

demonſtrated to be a third part of the priſm the baſe of which is the triangle ABC, and to which DEF is the oppoſite triangle; therefore the pyramid of which the baſe is the triangle ABC, and vertex the point D, is the third part of the priſm which has the ſame baſe, viz. the triangle ABC, and DEF is the oppoſite triangle. Q. E. D.

Cor. 1. From this it is manifeſt, that every pyramid is the third part of a priſm which has the ſame baſe, and is of an equal altitude with it; for if the baſe of the priſm be any other figure than a triangle, it may be divided into priſms having triangular baſes.

Cor. 2. Priſms of equal altitudes are to one another as their baſes; becauſe the pyramids upon the ſame baſes, and of the ſame altitude, are[c] to one another as their baſes.

c 6. 12.

PROP. VIII. THEOR.

SIMILAR pyramids having triangular baſes are one to another in the triplicate ratio of that of their homologous ſides.

Let the pyramids having the triangle ABC, DEF for their baſes, and the points G, H for their vertices, be ſimilar and ſimilarly ſituated; the pyramid ABCG has to the pyramid DEFH, the triplicate ratio of that which the ſide BC has to the homologous ſide EF.

Complete the parallelograms ABCM, GBCN, ABGK, and the ſolid parallelepiped BGML contained by theſe planes and
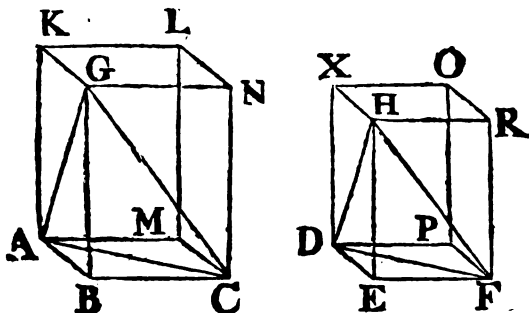
thoſe oppoſite to them: And, in like manner, complete the ſolid parallelepiped EHPO contained by the three parallelograms DEFP, HEFR, DEHX, and thoſe oppoſite to them: And be-
cauſe

caufe the pyramid ABCG is fimilar to the pyramid DEFH, the angle ABC is equal ª to the angle DEF, and the angle GBC to the angle HEF, and ABG to DEH : And AB is ᵇ to BC, as DE to EF ; that is, the fides about the equal angles are proportionals ; wherefore the parallelogram BM is fimilar to EP : For the fame reafon, the parallelogram BN is fimilar to ER, and BK to EX : Therefore the three parallelograms BM, BN, BK are fimilar to the three EP, ER, EX : But the three BM, BN, BK, are equal and fimilar ᶜ to the three which are oppofite to them, and the three EP, ER, EX equal and fimilar to the three oppofite to them : Wherefore the folids BGML, EHPO are contained by the fame number of fimilar planes ; and their folid angles are equal ᵈ ; and therefore the folid BGML is fimilar ᵉ to the folid EHPO : But fimilar folid parallelepipeds have the triplicate ᵉ ratio of that which their homologous fides have : Therefore the folid BGML has to the folid EHPO the triplicate ratio of that which the fide BC has to the homologous fide EF : But as the folid BGML is to the folid EHPO, fo is ᶠ the pyramid ABCG to the pyramid DEFH ; becaufe the pyramids are the fixth part of the folids, fince the prifm, which is the half ᵍ of the folid parallelepiped, is triple ʰ of the pyramid. Wherefore likewife the pyramid ABCG has to the pyramid DEFH, the triplicate ratio of that which BC has to the homologous fide EF. Q. E. D.

ª 11. def. 11.
ᵇ 1. def. 6.

ᶜ 24. 11.

ᵈ B. 11.

ᵉ 33. 11.

ᶠ 15. 5.

ᵍ 28. 11.
ʰ 7. 12.

Coʀ. From this it is evident, that fimilar pyramids which have multangular bafes, are likewife to one another in the triplicate ratio of their homologous fides : For they may be divided into fimilar pyramids having triangular bafes, becaufe the fimilar polygons, which are their bafes, may be divided into the fame number of fimilar triangles homologous to the whole polygons ; therefore as one of the triangular pyramids in the fiɪft multangular pyramid is to one of the triangular pyramids in the other, fo are all the triangular pyramids in the firft to all the triangular pyramids in the other ; that is, fo is the firft multangular pyramid to the other : But one triangular pyramid is to its fimilar triangular pyramid, in the triplicate ratio of their homologous fides ; and therefore the firft multangular pyramid has to the other, the triplicate ratio of that which one of the fides of the firft has to the homologous fide of the other.

See N.

P R O P.

P R O P. IX. T H E O R.

THE bafes and altitudes of equal pyramids having triangular bafes are reciprocally proportional : And triangular pyramids of which the bafes and altitudes are reciprocally proportional, are equal to one another.

Let the pyramids of which the triangles ABC, DEF are the bafes, and which have their vertices in the points G, H, be equal to one another : The bafes and altitudes of the pyramids ABCG, DEFH are reciprocally proportional, viz. the bafe ABC is to the bafe DEF, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG.

Complete the parallelograms AC, AG, GC, DF, DH, HF; and the folid parallelepipeds BGML, EHPO contained by
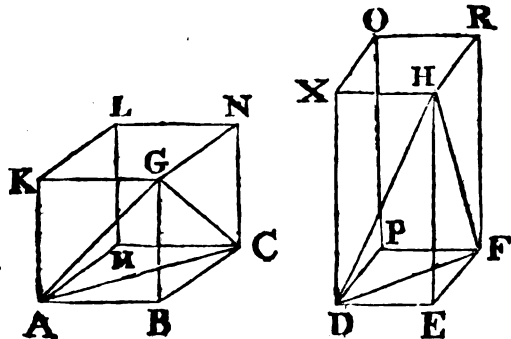
thefe planes and thofe oppofite to them : And becaufe the pyramid ABCG is equal to the pyramid DEFH, and that the folid BGML is fextuple of the pyramid ABCG, and the folid EHPO fextuple of the pyramid DEFH ; therefore the folid

a 1. Ax. 5. BGML is equal [a] to the folid EHPO : But the bafes and altitudes of equal folid parallelepipeds are reciprocally propor-

b 34. 11. tional [b] ; therefore as the bafe BM to the bafe EP, fo is the altitude of the folid EHPO to the altitude of the folid BGML :

c 15. 5. But as the bafe BM to the bafe EP, fo is [c] the triangle ABC to the triangle DEF ; therefore as the triangle ABC to the triangle DEF, fo is the altitude of the folid EHPO to the altitude of the folid BGML : But the altitude of the folid EHPO is the fame with the altitude of the pyramid DLFH ; and the altitude of the folid BGML is the fame with the altitude of the

pyramid

pyramid ABCG: Therefore, as the bafe ABC to the bafe DEF, so is the altitude of the pyramid DEFH to the altitude of the pyramid ABCG: Wherefore the bafes and altitudes of the pyramids ABCG, DEFH are reciprocally proportional.

Again, Let the bafes and altitudes of the pyramids ABCG, DEFH be reciprocally proportional, viz. the bafe ABC to the bafe DEF, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG: The pyramid ABCG is equal to the pyramid DEFH.

The fame conftruction being made, becaufe as the bafe ABC to the bafe DEF, so is the altitude of the pyramid DEFH to the altitude of the pyramid ABCG: And as the bafe ABC to the bafe DEF, so is the parallelogram BM to the parallelogram EP; therefore the parallelogram BM is to EP, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG: But the altitude of the pyramid DEFH is the fame with the altitude of the folid parallelepiped EHPO; and the altitude of the pyramid ABCG is the fame with the altitude of the folid parallelepiped BGML: As, therefore, the bafe BM to the bafe EP, so is the altitude of the folid parallelepiped EHPO to the altitude of the folid parallelepiped BGML. But folid parallelepipeds having their bafes and altitudes reciprocally proportional, are equal [b] to one another. Therefore the folid parallelepiped BGML is equal to the folid parallelepiped EHPO. And the pyramid ABCG is the fixth part of the folid BGML, and the pyramid DEFH the fixth part of the folid EHPO. Therefore the pyramid ABCG is equal to the pyramid DEFH. Therefore the bafes, &c. Q. E. D.

[b] 34. 11.

PROP. X. THEOR.

EVERY cone is the third part of a cylinder which has the fame bafe, and is of an equal altitude with it.

Let a cone have the fame bafe with a cylinder, viz. the circle ABCD, and the fame altitude. The cone is the third part of the cylinder; that is, the cylinder is triple of the cone.

If the cylinder be not triple of the cone, it muft either be greater than the triple, or lefs than it. Firft, Let it be greater than the triple; and defcribe the fquare ABCD in the circle; this fquare is greater than the half of the circle ABCD [*].

Upon

[*] As was fhewn in prop. 2. of this book.

Upon the fquare ABCD erect a prifm of the fame altitude with the cylinder; this prifm is greater than half of the cylinder; becaufe if a fquare be defcribed about the circle, and a prifm erected upon the fquare, of the fame altitude with the cylinder, the infcribed fquare is half of that circumfcribed; and upon thefe fquare bafes are erected folid parallelepipeds, viz. the prifms, of the fame altitude; therefore the prifm upon the fquare ABCD is the half of the prifm upon the fquare defcribed about the circle: Becaufe they are to one another as their

a 32. 11. bafes [a]: And the cylinder is lefs than the prifm upon the fquare defcribed about the circle ABCD: Therefore the prifm upon the fquare ABCD of the fame altitude with the cylinder, is greater than half of the cylinder. Bifect the circumferences AB, BC, CD, DA in the points E, F, G, H; and join AE, EB, BF, FC, CG, GD, DH, HA: Then, each of the triangles AEB, BFC, CGD, DHA is greater than the half of the feg-ment of the circle in which it ftands, as was fhewn in prop. 2. of this book. Erect prifms upon each of thefe triangles of the fame altitude with the cylinder; each of thefe prifms is greater than half of the feg-ment of the cylinder in which it is; becaufe if, through the points E, F, G, H, parallels be drawn to AB, BC, CD, DA, and parallelograms be completed upon the fame AB, BC, CD, DA, and folid parallelepipeds
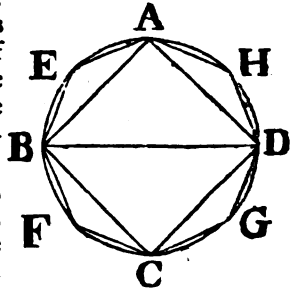
be erected upon the parallelograms; the prifms upon the triangles AEB, BFC, CGD, DHA are the halves of the folid

b 2. Cor. parallelepipeds [b]. And the fegments of the cylinder which are
7. 12. upon the fegments of the circle cut off by AB, BC, CD, DA, are lefs than the folid parallelepipeds which contain them. Therefore the prifms upon the triangles AEB, BFC, CGD, DHA, are greater than half of the fegments of the cylinder in which they are; therefore, if each of the circumferences be di-vided into two equal parts, and ftraight lines be drawn from the points of divifion to the extremities of the circumferences, and upon the triangles thus made, prifms be erected of the fame altitude with the cylinder, and fo on, there muft at length re-

c Lemma. main fome fegments of the cylinder which together are lefs [c] than the excefs of the cylinder above the triple of the cone. Let them be thofe upon the fegments of the circle AE, EB, BF, FC,

FC, CG, GD, DH, HA. Therefore the reſt of the cylin-
der, that is, the priſm of which the baſe is the polygon
AEBFCGDH, and of which the altitude is the ſame with that
of the cylinder, is greater than the triple of the cone : But this
priſm is triple ^d of the pyramid upon the ſame baſe, of which
the vertex is the ſame with the vertex of the cone ; therefore
the pyramid upon the baſe AEBFCGDH, having the ſame
vertex with the cone, is greater than the cone, of which the
baſe is the circle ABCD : But it is alſo leſs, for the pyramid is
contained within the cone ; which is impoſſible. Nor can the
cylinder be leſs than the triple of the cone. Let it be leſs, if
poſſible : Therefore, inverſely, the cone is greater than the third
part of the cylinder. In the circle ABCD deſcribe a ſquare ;
this ſquare is greater than the half of the circle : And upon the
ſquare ABCD erect a pyramid having the ſame vertex with the
cone ; this pyramid is greater than the half of the cone ; becauſe,
as was before demonſtrated, if a ſquare be deſcribed about the

circle, the ſquare ABCD is the
half of it ; and if, upon theſe ſquares
there be erected ſolid parallelepi-
peds of the ſame altitude with the
cone, which are alſo priſms, the
priſm upon the ſquare ABCD
ſhall be the half of that which is
upon the ſquare deſcribed about the
circle ; for they are to one another
as their baſes ^a ; as are alſo the third
parts of them : Therefore the py-
ramid, the baſe of which is the

a 32. 11.

ſquare ABCD, is half of the pyramid upon the ſquare de-
ſcribed about the circle : But this laſt pyramid is greater than
the cone which it contains ; therefore the pyramid upon the
ſquare ABCD having the ſame vertex with the cone, is great-
er than the half of the cone. Biſect the circumferences AB,
BC, CD, DA in the points E, F, G, H, and join AE, EB,
BF, FC, CG, GD, DH, HA : Therefore each of the triangles
AEB, BFC, CGD, DHA is greater than half of the ſegment
of the circle in which it is : Upon each of theſe triangles erect
pyramids having the ſame vertex with the cone. Therefore each
of theſe pyramids is greater than the half of the ſegment of
the cone in which it is, as before was demonſtrated of the
priſms and ſegments of the cylinder ; and thus dividing each
of the circumferences into two equal parts, and joining the
points
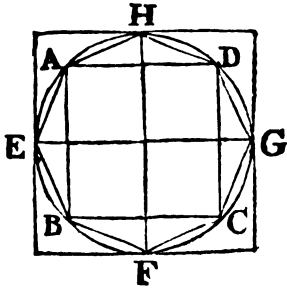
points of divifion and their extremities by ftraight lines, and upon the triangles erecting pyramids having their vertices the fame with that of the cone, and fo on, there muft at length remain fome fegments of the cone, which together fhall be lefs than the excefs of the cone, above the third part of the cylinder. Let thefe be the fegments upon AE, EB, BF, FC, CG, GD, DH, HA. Therefore the reft of the cone, that is, the pyramid, of which the bafe is the polygon AEBFCGDH, and of which the vertex is the fame with that of the cone, is greater than the third part of the cylinder. But this pyramid is the third part of the prifm upon the fame bafe AEBFCGDH, and of the fame altitude with the cylinder. Therefore this prifm is greater than the cylinder of which the

bafe is the circle ABCD. But it is alfo lefs, for it is contained within the cylinder; which is impoffible. Therefore the cylinder is not lefs than the triple of the cone. And it has been demonftrated that neither is it greater than the triple. Therefore the cylinder is triple of the cone, or, the cone is the third part of the cylinder. Wherefore every cone, &c. Q. E. D.
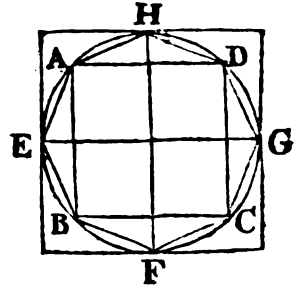
PROP. XI. THEOR.

CONES and cylinders of the fame altitude, are to one another as their bafes.

Let the cones and cylinders, of which the bafes are the circles ABCD, EFGH, and the axes KL, MN, and AC, EG the diameters of their bafes, be of the fame altitude. As the circle ABCD to the circle EFGH, fo is the cone AL to the cone EN.

If it be not fo, let the circle ABCD be to the circle EFGH, as the cone AL to fome folid either lefs than the cone EN, or greater than it. Firft, let it be to a folid lefs than EN, viz. to the folid X; and let Z be the folid which is equal to the excefs of the cone EN above the folid X; therefore the cone EN is equal to the folids X, Z together. In the circle EFGH defcribe the fquare EFGH, therefore this fquare is greater than the half of the circle: Upon the fquare EFGH erect a pyramid of the fame altitude with the cone; this pyramid is greater than half of the cone. For, if a fquare be defcribed about the circle, and a pyramid be erected upon it, having

ving the fame vertex with the cone *, the pyramid infcribed Book XII.
in the cone is half of the pyramid circumfcribed about it, be-
caufe they are to one another as their bafes ª : But the cone a 6. 12.
is lefs than the circumfcribed pyramid ; therefore the pyramid of
which the bafe is the fquare EFGH, and its vertex the fame
with that of the cone, is greater than half of the cone : Divide
the circumferences EF, FG, GH, HE, each into two equal
parts in the points O, P, R, S, and join EO, OF, FP, PG,
GR, RH, HS, SE : Therefore each of the triangles EOF,
FPG, GRH, HSE is greater than balf of the fegment of the
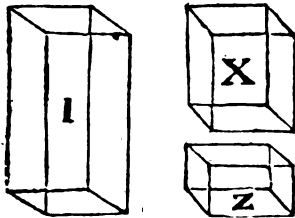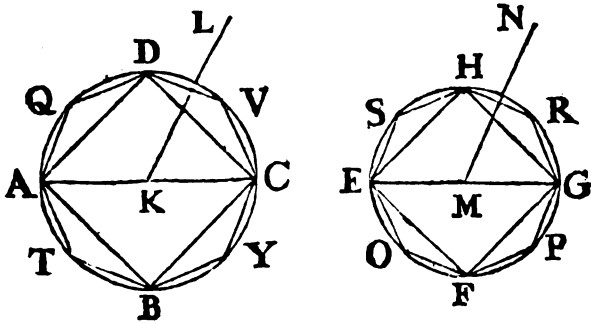
circle in which it is : Upon each of thefe triangles erect a pyra-
mid having the fame vertex with the cone; each of thefe py-
ramids is greater than the half of the fegm nt of the cone in
which it is : And thus dividing each of thefe circumferences
into two equal parts, and from the points of divifion drawing
ftraight lines to the extremities of the circumferences, and up-
on each of the triangles thus made erecting pyramids having the
fame vertex with the cone, and fo on, there muft at length
remain fome fegments of the cone which are together lefs ᵇ b Lemma,
than the folid Z : Let thefe be the fegments upon EO, OF, FP,
<div align="center">S</div> PG,

PG, GR, RH, HS, SE : Therefore the remainder of the cone, viz. the pyramid of which the bafe is the polygon EOFPGRHS, and its vertex the fame with that of the cone, is greater than the folid X : In the circle ABCD defcribe the polygon ATBYCVDQ fimilar to the polygon EOFPGRHS, and upon it erect a pyramid having the fame vertex with the cone AL :
And becaufe as the fquare of AC is to the fquare of EG, fo ᵃ is the polygon ATBYCVDQ to the polygon EOFPGRHS ; and
as the fquare of AC to the fquare of EG, fo is ᵇ the circle
ABCD to the circle EFGH ; therefore the circle ABCD is ᶜ to the circle EFGH, as the polygon ATBYCVDQ to the poly-

gon EOFPGRHS : But as the circle ABCD to the circle EFGH, fo is the cone AL to the folid X ; and as the polygon
ATBYCVDQ to the polygon EOFPGRHS, fo is ᵈ the pyramid of which the bafe is the firft of thefe polygons, and vertex L, to the pyramid of which the bafe is the other polygon, and its vertex N : Therefore, as the cone AL to the folid X, fo is the pyramid of which the bafe is the polygon ATBYCVDQ, and vertex L, to the pyramid the bafe of which is the polygon EOFPGRHS, and vertex N : But the cone AL is greater than
the pyramid contained in it ; therefore the folid X is greater ᵉ than the pyramid in the cone EN ; But it is lefs, as was fhown, **which**
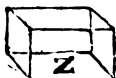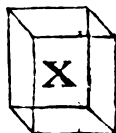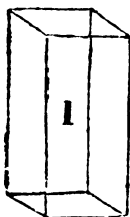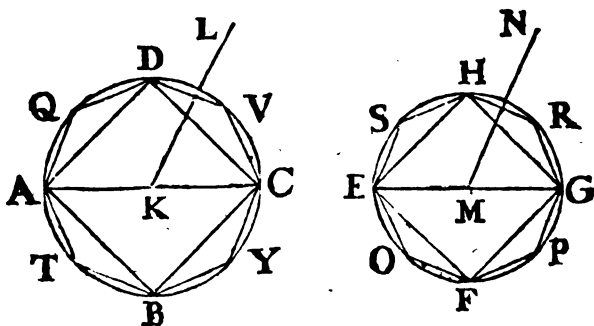
which is abfurd: Therefore the circle ABCD is not to the circle EFGH, as the cone AL to any folid which is lefs than the cone EN. In the fame manner it may be demonftrated that the circle EFGH is not to' the circle ABCD, as the cone EN to any folid lefs than the cone AL. Nor can the circle ABCD be to the circle EFGH, as the cone AL to any folid greater than the cone EN: For, if it be poffible, let it be fo to the folid I, which is greater than the cone EN: Therefore, by inverfion, as the circle EFGH to the circle ABCD, fo is the folid I to the cone AL: But as the folid I to the cone AL, fo is the cone EN to fome folid, which muft be lefs ᵃ than the cone a 14. 5. AL, becaufe the folid I is greater than the cone EN: Therefore, as the circle EFGH is to the circle ABCD, fo is the cone EN to a folid lefs than the cone AL, which was fhewn to be impoffible: Therefore the circle ABCD is not to the circle EFGH, as the cone AL is to any folid greater than the cone EN: And it has been demonftrated that neither is the circle ABCD to the circle EFGH, as the cone AL to any folid lefs than the cone EN: Therefore the circle ABCD is to the circle EFGH, as the cone AL to the cone EN: But as the cone is to the cone, fo ᵇ is the cylinder to the cylinder, becaufe the cy- b 15. 5. linders are triple ᶜ of the cones, each to each. Therefore, as c 10. 12. the circle ABCD to the circle EFGH, fo are the cylinders upon them of the fame altitude. Wherefore cones and cylinders of the fame altitude are to one another as their bafes. Q. E. D.

PROP. XII. THEOR.

SIMILAR cones and cylinders have to one another See N. the triplicate ratio of that which the diameters of their bafes have.

Let the cones and cylinders of which the bafes are the circles ABCD, EFGH, and the diameters of the bafes AC, EG, and KL, MN the axes of the cones or cylinders, be fimilar: The cone of which the bafe is the circle ABCD, and vertex the point L, has to the cone of which the bafe is the circle EFGH, and vertex N, the triplicate ratio of that which AC has to EG.
For, if the cone ABCDL has not to the cone EFGHN the triplicate ratio of that which AC has to EG, the cone ABCDL fhall have the triplicate of that ratio to fome folid which is lefs

or

or greater than the cone EFGHN. Firſt, let it have it to a leſs, viz. to the ſolid X : Make the ſame conſtruction as in the preceding propoſition, and it may be demonſtrated the very ſame way as in that propoſition, that the pyramid of which the baſe is the polygon EOFPGRHS, and vertex N, is greater than the ſolid X. Deſcribe alſo in the circle ABCD the polygon ATBYCVDQ fimilar to the polygon EOFPGRHS, upon which erect a pyramid having the ſame vertex with the cone ; and let LAQ be one of the triangles containing the pyramid upon the polygon ATBYCVDQ the vertex of which is L; and let NES be one of the triangles containing the pyramid upon the
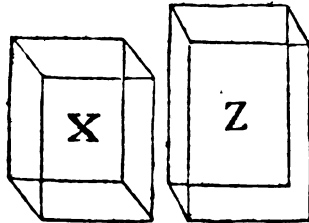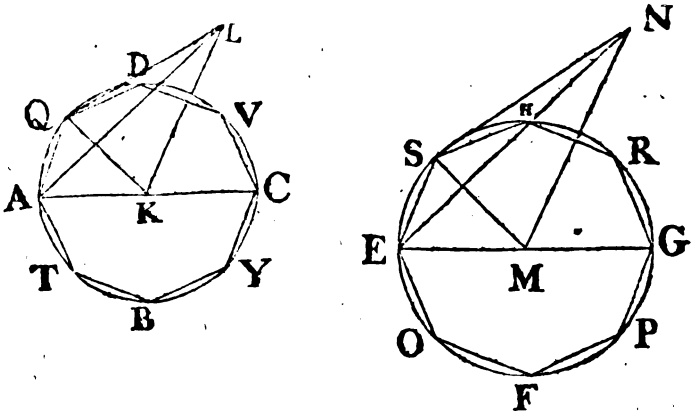
polygon EOFPGRHS of which the vertex is N ; and join KQ, MS : Becauſe then the cone ABCDL is fimilar to the cone EFGHN, AC is [a] to EG, as the axis KL to the axis MN ; and as AC to EG, ſo [b] is AK to EM ; therefore as AK to EM, ſo is KL to MN ; and, alternately, AK to KL, as EM to MN : And the right angles AKL, EMN are equal ; therefore the ſides about theſe equal angles being proportionals, the triangle AKL is fimilar [c] to the triangle EMN. Again, becauſe AK is to KQ, as EM to MS, and that theſe ſides are
about

a 24. def.
11.
b 15. 5.

c 6. 6.

about equal angles AKQ, EMS, becaufe thefe angles are, each of them, the fame part of four right angles at the centres K, M; therefore the triangle AKQ is fimilar [a] to the triangle EMS: And becaufe it has been fhown that as AK to KL, fo is EM to MN, and that AK is equal to KQ; and EM to MS, as QK to KL, fo is SM to MN; and therefore the fides about the right angles QKL, SMN being proportionals, the triangle LKQ is fimilar to the triangle NMS: And becaufe of the fimilarity of the triangles AKL, EMN, as LA is to AK, fo is NE to EM; and by the fimilarity of the triangles AKQ, EMS, as KA to AQ, fo ME to ES; ex aequali [b], LA is to AQ, as NE to ES. Again, becaufe of the fimilarity of the triangles LQK, NSM, as LQ to QK, fo NS to SM; and from the fimilarity of the triangles KAQ, MES, as KQ to QA, fo MS to SE; ex aequali [b], LQ is to QA, as NS to SE: And it was proved that QA is to AL, as SE to EN; therefore, again, ex aequali, as QL to LA, fo is SN to NE: Wherefore the triangles LQA, NSE, having the fides about all their angles proportionals, are equiangular [c] and fimilar to one another: And therefore the pyramid of which the bafe is the triangle AKQ, and vertex L, is fimilar to the pyramid the bafe of which is the triangle EMS, and vertex N, becaufe their folid angles are equal [d] to one another, and they are contained by the fame number of fimilar planes: But fimilar pyramids which have triangular bafes have to one another the triplicate ratio of that which their homologous fides have; therefore [e] the pyramid AKQL has to the pyramid EMSN the triplicate ratio of that which AK has to EM. In the fame manner, if ftraight lines be drawn from the points D, V, C, Y, B, T to K, and from the points H, R, G, P, F, O to M, and pyramids be erected upon the triangles having the fame vertices with the cones, it may be demonftrated that each pyramid in the firft cone has to each in the other, taking them in the fame order, the triplicate ratio of that which the fide AK has to the fide EM; that is, which AC has to EG: But as one antecedent to its confequent, fo are all the antecedents to all the confequents [f]; therefore as the pyramid AKQL to the pyramid EMSN, fo is the whole pyramid the bafe of which is the polygon DQATBYCV, and vertex L, to the whole pyramid of which the bafe is the polygon HSEOFPGR, and vertex N. Wherefore alfo the firft of thefe two laft named pyramids has to the other the triplicate ratio of that which AC has to EG. But, by the hypothefis, the cone of which the bafe is the circle ABCD, and vertex L has to the folid X, the triplicate ratio of that which AC has to EG; therefore, as the cone of

S 3   which

which the bafe is the circle ABCD, and vertex L, is to the
folid X, fo is the pyramid the bafe of which is the polygon
DQATBYCV, and vertex L, to the pyramid the bafe of which
is the polygon HSEOFPGR and vertex N: But the faid cone
is greater than the pyramid contained in it, therefore the folid
a 14. 5.   X is greater ª than the pyramid, the bafe of which is the poly-
gon HSEOFPGR, and vertex N; but it is alfo lefs; which is
impoffible: Therefore the cone of which the bafe is the circle
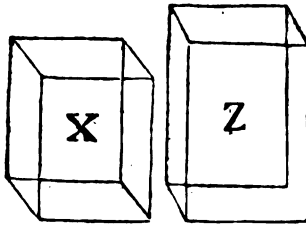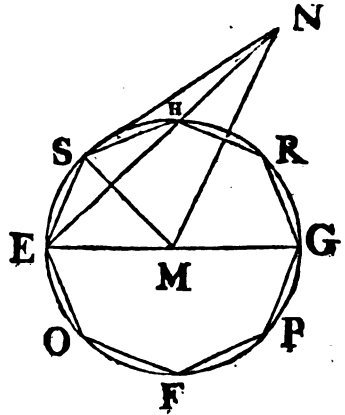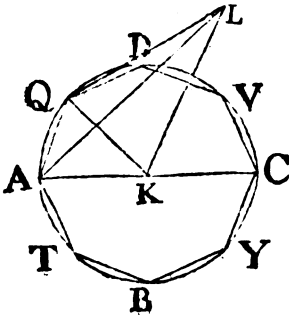
ABCD, and vertex L, has not to any folid which is lefs than the
cone of which the bafe is the circle ÉFGH and vertex N, the tri-
plicate ratio of that which AC has to EG. In the fame manner
it may be demonftrated that neither has the cone EFGHN to
any folid which is lefs than the cone ABCDL, the triplicate
ratio of that which EG has to AC. Nor can the cone ABCDL
have to any folid which is greater than the cone EFGHN, the
triplicate ratio of that which AC has to EG: For, if it be pof-
fible, let it have it to a greater, viz. to the folid Z: Therefore,
inverfely, the folid Z has to the cone ABCDL, the triplicate
ratio of that which EG has to AC: But as the folid Z is to
                                                              the

Book XII.

the cone ABCDL, fo is the cone EFGHN to fome folid, which muft be lefs [a] than the cone ABCDL, becaufe the folid Z is greater than the cone EFGHN: Therefore the cone EFGHN [a] 14. 5. has to a folid which is lefs than the cone ABCDL, the triplicate ratio of that which EG has to AC, which was demonftrated to be impoffible: Therefore the cone ABCDL has not to any folid greater than the cone EFGHN, the triplicate ratio of that which AC has to EG; and it was demonftrated that it could not have that ratio to any folid lefs than the cone EFGHN: Therefore the cone ABCDL has to the cone EFGHN, the triplicate ratio of that which AC has to EG: But as the cone is to the cone, fo [b] the cylinder to the cylinder; for every [b] 15. 5. cone is the third part of the cylinder upon the fame bafe, and of the fame altitude: Therefore alfo the cylinder has to the cylinder, the triplicate ratio of that which AC has to EG. Wherefore fimilar cones, &c.   Q. E. D.

PROP. XIII.   THEOR.

IF a cylinder be cut by a plane parallel to its oppofite See N. planes, or bafes; it divides the cylinder into two cylinders, one of which is to the other as the axis of the firft to the axis of the other.

Let the cylinder AD be cut by the plane GH parallel to the oppofite planes AB, CD, meeting the axis EF in the point K, and let the line GH be the common fection of the plane GH and the furface of the cylinder AD: Let AEFC be the parallelogram, in any pofition of it, by the revolution of which about the ftraight line EF the cylinder AD is defcribed; and let GK be the common fection of the plane GH, and the plane AEFC: And becaufe the parallel planes AB, GH are cut by the plane AEKG, AE, KG, their common fections with it, are parallel [a]; wherefore AK is a parallelogram, and GK equal to EA the ftraight line from the centre of the circle AB: For the fame reafon, each of the ftraight lines
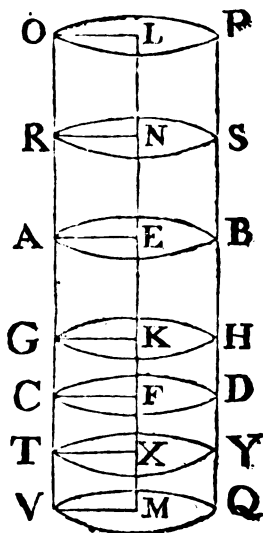
[a] 16. 11.

S 4

drawn

Book XII. drawn from the point **K** to the line **GH** may be proved to be
equal to those which are drawn from the centre of the circle
AB to its circumference, and are therefore all equal to one an-
a 15. def. I. other. Therefore the line GH is the circumference of a circle [a]
of which the centre is the point **K** : Therefore the plane **GH**
divides the cylinder AD into the cylinders AH, GD ; for they
are the same which would be described by the revolution of the
parallelograms AK, GF about the straight lines EK, KF : And
it is to be shown that the cylinder AH is to the cylinder HC, as
the axis EK to the axis KF

Produce the axis EF both ways ; and take any number of
straight lines EN, NL, each equal to EK ; and any number
FX, XM, each equal to FK ; and let
planes parallel to AB. CD pass through
the points L, N, X, M : Therefore
the common sections of these planes
with the cylinder produced are circles
the centres of which are the points
L, N, X, M, as was proved of the
plane GH ; and these planes cut off
the cylinders, PR, RB, DT, TQ :
And because the axes LN, NE, EK
are all equal ; therefore the cylinders
b 11. 12. PR, RB, BG are [b] to one another as
their bases ; but their bases are equal,
and therefore the cylinders PR, RB,
BG are equal : And because the axes
LN, NE, EK are equal to one an-
other, as also the cylinders PR, RB,
BG, and that there are as many axes
as cylinders ; therefore, whatever mul-
tiple the axis KL is of the axis KE,
the same multiple is the cylinder PG
of the cylinder GB : For the same reason, whatever multiple
the axis MK is of the axis KF, the same multiple is the cy-
linder QG of the cylinder GD : And if the axis KL be equal to
the axis KM, the cylinder PG is equal to the cylinder GQ ; and
if the axis KL be greater than the axis KM, the cylinder PG is
greater than the cylinder GQ ; and if less, less : Since there-
fore there are four magnitudes, viz. the axes EK, KF, and
the cylinders BG, GD, and that of the axis EK and cylinder
BG there has been taken any equimultiples whatever, viz. the
axis
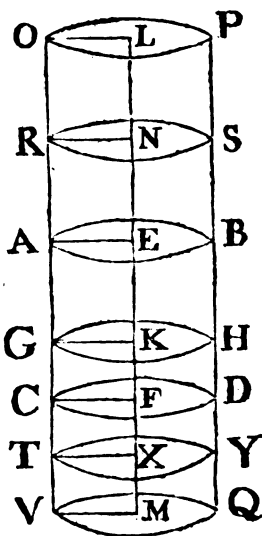
axis KL and cylinder PG; and of the axis KF and cylinder
GD, any equimultiples whatever, viz the axis KM and cylin-
der GQ; and it has been demonftrated, if the axis KL be
greater than the axis KM, the cylinder PG is greater than the
cylinder GQ; and if equal, equal; and if lefs, lefs: Therefore
d the axis EK is to the axis KF, as the cylinder BG to the cylin- d 5. def.
der GD. Wherefore, if a cylinder, &c. Q. E. D.   5.

PROP. XIV. THEOR.

CONES and cylinders upon equal bafes are to one an-
other as their altitudes.

Let the cylinders EB, FD be upon the equal bafes AB, CD:
As the cylinder EB to the cylinder FD, fo is the axis GH to
the axis KL.

Produce the axis KL to the point N, and make LN equal
to the axis GH, and let CM be a cylinder of which the bafe is
CD, and axis LN; and becaufe the cylinders EB, CM have
the fame altitude, they are to one another as their bafes a: But a 11. 12.
their bafes are equal, therefore alfo the cylinders EB, CM are

equal. And becaufe the cylin-
der FM is cut by the plane
CD parallel to its oppofite
planes, as the cylinder CM to
the cylinder FD, fo is b the
axis LN to the axis KL. But
the cylinder CM is equal to
the cylinder EB, and the axis
LN to the axis GH: There-
fore as the cylinder EB to the
cylinder FD, fo is the axis
GH to the axis KL: And as

b 13. 12.

the cylinder EB to the cylinder FD, fo is c the cone ABG to the   c 15. 5.
cone CDK, becaufe the cylinders are triple d of the cones: d 10. 12.
Therefore alfo the axis GH is to the axis KL, as the cone ABG
to the cone CDK, and the cylinder EB to the cylinder FD.
Wherefore cones, &c. Q. E. D.
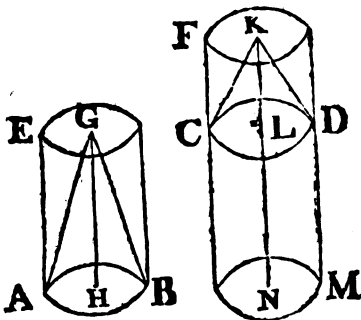
PROP.

PROP. XV. THEOR.

See N.

THE bafes and altitudes of equal cones and cylinders, are reciprocally proportional; and if the bafes and altitudes be reciprocally proportional, the cones and cylinders are equal to one another.

Let the circles ABCD, EFGH, tho diameters of which are AC, EG, be the bafes, and KL, MN the axes, as alfo the altitudes, of equal cones and cylinders; and let ALC, ENG be the cones, and AX, EO the cylinders: The bafes and altitudes of the cylinders AX, EO are reciprocally proportional; that is, as the bafe ABCD to the bafe EFGH, fo is the altitude MN to the altitude KL.

Either the altitude MN is equal to the altitude KL, or thefe altitudes are not equal. Firft, let them be equal; and the cylinders AX, EO being alfo equal, and cones and cylinders of the fame altitude being to one another as their bafes [a], therefore the bafe ABCD is equal [b] to the bafe EFGH; and as the bafe ABCD is to the bafe EFGH, fo is the altitude MN to the altitude KL.

a II. 12.
b A. 5.

But let the altitudes KL, MN be unequal, and MN the greater of the two, and from MN take MP equal to KL, and, through the point P, cut the cylinder EO by the plane TYS parallel to the op-

pofite planes of the circles EFGH, RO; therefore the common fection of the plane TYS and the cylinder EO is a circle, and confequently ES is a cylinder, the bafe of which is the circle EFGH, and altitude MP: And becaufe the cylinder AX is equal to the cylinder EO, as AX is to the cylinder ES, fo [c] is the cylinder EO to the fame ES. But as the cylinder AX to the cylinder ES, fo [a] is the bafe ABCD to the bafe EFGH; for the cylinders AX, ES are of the fame altitude; and as the cylinder EO to the cylinder ES, fo [d] is the altitude MN to the altitude MP, becaufe the cylinder EO is cut by the plane
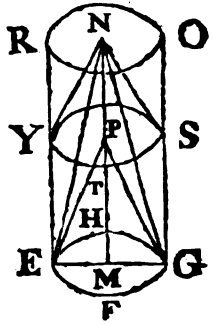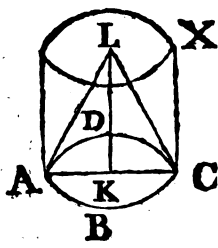
c 7. 5.

d 13. 12.

TYS

TYS parallel to its oppofite planes. Therefore as the bafe ABCD to the bafe EFGH, fo is the altitude MN to the altitude MP: But MP is equal to the altitude KL; wherefore as the bafe ABCD to the bafe EFGH, fo is the altitude MN to the altitude KL; that is, the bafes and altitudes of the equal cylinders AX, EO are reciprocally proportional.

But let the bafes and altitudes of the cylinders AX, EO, be reciprocally proportional, viz. the bafe ABCD to the bafe EFGH, as the altitude MN to the altitude KL: The cylinder AX is equal to the cylinder EO.

First, let the bafe ABCD be equal to the bafe EFGH; then, becaufe as the bafe ABCD is to the bafe EFGH, fo is the altitude MN to the altitude KL; MN is equal ^b to KL, and therefore the cylinder AX is equal ^a to the cylinder EO.

But let the bafes ABCD, EFGH be unequal, and let ABCD be the greater; and becaufe, as ABCD is to the bafe EFGH, fo is the altitude MN to the altitude KL; therefore MN is greater ^b than KL. Then, the fame conftruction being made as before, becaufe as the bafe ABCD to the bafe EFGH, fo is the altitude MN to the altitude KL; and becaufe the altitude KL is equal to the altitude MP; therefore the bafe ABCD is ^a to the bafe EFGH, as the cylinder AX to the cylinder ES; and as the altitude MN to the altitude MP or KL, fo is the cylinder EO to the cylinder ES: Therefore the cylinder AX is to the cylinder ES, as the cylinder EO is to the fame ES: Whence the cylinder AX is equal to the cylinder EO: And the fame reafoning holds in cones. Q. E. D.

PROP. XVI. PROB.

TO defcribe in the greater of two circles that have the fame centre, a polygon of an even number of equal fides, that fhall not meet the leffer circle.

Let ABCD, EFGH be two given circles having the fame centre K: It is required to infcribe in the greater circle ABCD a polygon of an even number of equal fides, that fhall not meet the leffer circle.

Through the centre K draw the ftraight line BD, and from the point G, where it meets the circumference of the leffer circle

Book XII. circle, draw GA at right angles to BD, and produce it to C;

ᵃ 16. 3. therefore AC touches ᵃ the circle EFGH: Then, if the circum-

ᵇ Lemma. ference BAD be bifected, and the half of it be again bifected, and so on, there must at length remain a circumference less ᵇ than AD: Let this be LD; and from the point L draw LM per-pendicular to BD, and produce it to N; and join LD, DN. Therefore LD is equal to DN; and because LN is parallel to AC, and that AC touches the circle EFGH; therefore LN does not meet the circle EFGH: And much less shall the straight lines LD, DN meet the circle EFGH:
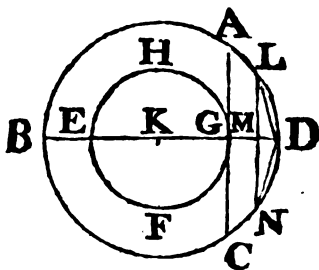
So that if straight lines equal to LD be applied in the circle ABCD from the point L around to N, there shall be described in the circle a polygon of an even number of equal sides not meeting the lesser circle.   Which was to be done.

L E M M A   II.

IF two trapeziums ABCD, EFGH be inscribed in the circles, the centres of which are the points K, L; and if the sides AB, DC be parallel, as also EF, HG; and the other four sides AD, BC, EH, FG be all equal to one another; but the side AB greater than EF, and DC greater than HG.   The straight line KA from the centre of the circle in which the great-er sides are, is greater than the straight line LE drawn from the centre to the circumference of the other circle.

If it be possible, let KA be not greater than LE; then KA must be either equal to it, or less.   First, let KA be equal to LE: Therefore, because in two equal circles, AD, BC in the one are equal to EH, FG in the other, the circumferences

ᵃ 28. 3. AD, BC are equal ᵃ to the circumferences EH, FG; but be-cause the straight lines AB, DC are respectively greater than EF, GH, the circumferences AB, DC are greater than EF, HG: Therefore the whole circumference ABCD is greater than the whole EFGH; but it is also equal to it, which is
impossible:

impoſſible: Therefore the ſtraight line KA is not equal to LE.

But let KA be leſs than LE, and make LM equal to KA, and from the centre L, and diſtance LM deſcribe the circle MNOP, meeting the ſtraight lines LE, LF, LG, LH, in M, N, O, P; and join MN, NO, OP, PM which are reſpective-ly parallel [a] to, and leſs than EF, FG, GH, HE: Then, be-cauſe EH is greater than MP, AD is greater than MP; and

a 2. 6.
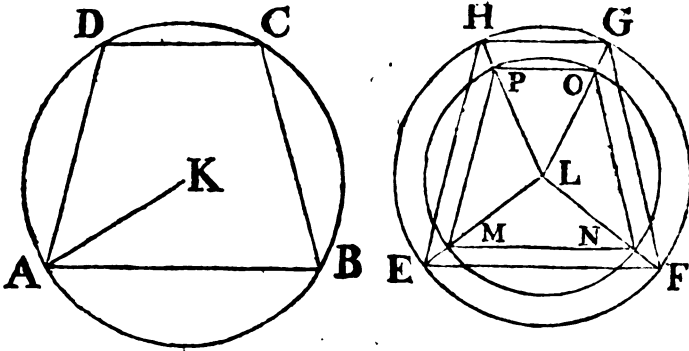
the circles ABCD, MNOP are equal; therefore the circum-ference AD is greater than MP; for the ſame reaſon, the cir-cumference BC is greater than NO; and becauſe the ſtraight line AB is greater than EF which is greater than MN, much more is AB greater than MN: Therefore the circumference AB is greater than MN; and, for the ſame reaſon, the circum-ference DC is greater than PO: Therefore the whole circum-ference ABCD is greater than the whole MNOP; but it is like-wiſe equal to it, which is impoſſible: Therefore KA is not leſs than LE; nor is it equal to it; the ſtraight line KA muſt therefore be greater than LE. Q. E. D.

Cor. And if there be an iſoſceles triangle the ſides of which are equal to AD, BC, but its baſe leſs than AB the greater of the two ſides AB, DC; the ſtraight line KA may, in the ſame manner, be demonſtrated to be greater than the ſtraight line drawn from the centre to the circumference of the circle de-ſcribed about the triangle.

· PROP.

PROP. XVII. PROB.

TO defcribe in the greater of two fpheres which have the fame centre, a folid polyhedron, the fuperficies of which fhall not meet the leffer fphere.

Let there be two fpheres about the fame centre A; it is required to defcribe in the greater a folid polyhedron, the fuperficies of which fhall not meet the leffer fphere.

Let the fpheres be cut by a plane paffing through the centre; the common fections of it with the fpheres fhall be circles; becaufe the fphere is defcribed by the revolution of a femicircle about the diameter remaining unmoveable; fo that in whatever pofition the femicircle be conceived, the common fection of the plane in which it is with the fuperficies of the fphere is the circumference of a circle; and this is a great circle of the fpheie, becaufe the diameter of the fphere, which is likewife

the diameter of the circle, is greater ^{a} than any ftraight line in the circle or fphere: Let then the circle made by the fection of the plane with the greater fphere be BCDE, and with the leffer fphere be FGH; and draw the two diameters BD, CE at right angles to one another: And in BCDE, the greater of

the two circles, defcribe ^{b} a polygon of an even number of equal fides not meeting the leffer circle FGH; and let its fides, in BE, the fourth part of the circle, be BK, KL, LM, ME; join KA and produce it to N; and from A draw AX at right angles to the plane of the circle BCDE meeting the fuperficies of the fphere in the point X; and let planes pafs through AX and each of the ftraight lines BD, KN, which, from what has been faid, fhall produce great circles on the fuperficies of the fphere, and let BXD, KXN be the femicircles thus made upon the diameters BD, KN: Therefore, becaufe XA is at right angles to the plane of the circle BCDE, every plane which

paffes through XA is at right ^{c} angles to the plane of the circle BCDE; wherefore the femicircles BXD, KXN are at right angles to that plane: And becaufe the femicircles BED, BXD, KXN, upon the equal diameters BD KN, are equal to one another, their halves BE, BX, KX, are equal to one another: Therefore, as many fides of the polygon as are in BE, fo many there are in BX, KX equal to the fides BK, KL, LM, ME: Let thefe polygons be defcribed, and their fides be BO, OP, PR, RX; KS, ST, TY, YX, and join
OS,

OS, PT, RY; and from the points O, S draw OV, SQ perpendiculars to AB, AK: And becaufe the plane BOXD is at right angles to the plane BCDE, and in one of them BOXD, OV is drawn perpendicular to AB the common fection of the planes, therefore OV is perpendicular ᵃ to the plane BCDE: For the fame reafon SQ is perpendicular to the fame plane, becaufe the plane KSXN is at right angles to the plane BCDE. Join VQ; and, becaufe in the equal femicircles BXD, KXN the

a 4. def. 11.
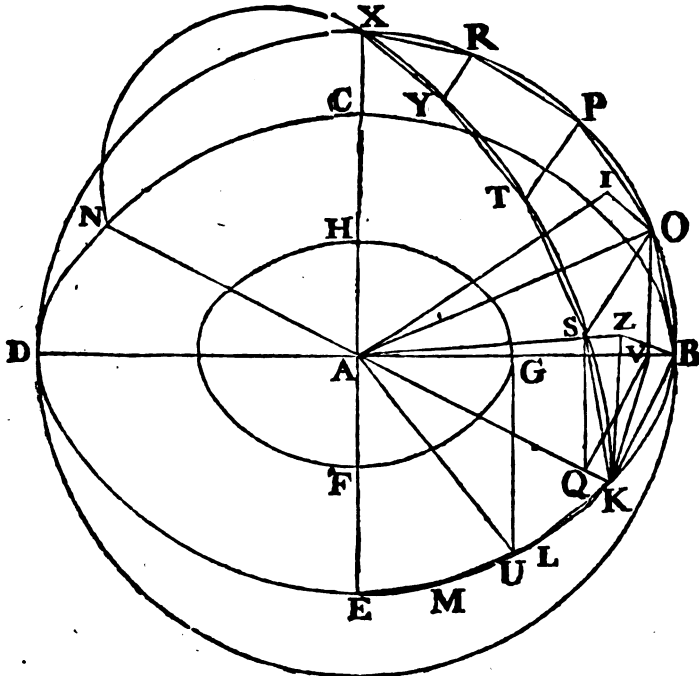
circumferences BO, KS are equal, and OV, SQ are perpendicular to their diameters, therefore ᵈ OV is equal to SQ, and BV equal to KQ: But the whole BA is equal to the whole KA, therefore the remainder VA is equal to the remainder QA: As therefore BV is to VA, fo is KQ to QA, wherefore VQ is parallel ᵉ to BK: And becaufe OV, SQ are each of them at right angles to the plane of the circle BCDE, OV is parallel ᶠ to SQ; and it has been proved that it is alfo equal to it; therefore QV, SO are equal and parallel ᵍ: And becaufe QV is parallel to SO, and alfo to KB; OS is parallel ʰ to BK; and therefore BO, KS which join them are in the fame plane ʰ

in

d 26. 1.

e 2. 6.

f 6. 11.

g 33. 1.

h 9. 11.

in which thefe parallels are, and the quadrilateral figure **KBOS**
is in one plane: And if PB, TK be joined, and perpendiculars
be drawn from the points P, T to the ftraight lines AB, AK,
it may be demonftrated that TP is parallel to KB in the very
fame way that SO was fhown to be parallel to the fame KB;
a 9. 11. wherefore ª TP is parallel to SO, and the quadrilateral figure
SOPT is in one plane: For the fame reafon, the quadrilateral
b 2. 11. TPRY is in one plane: And the figure YRX is alfo in one plane ᵇ.
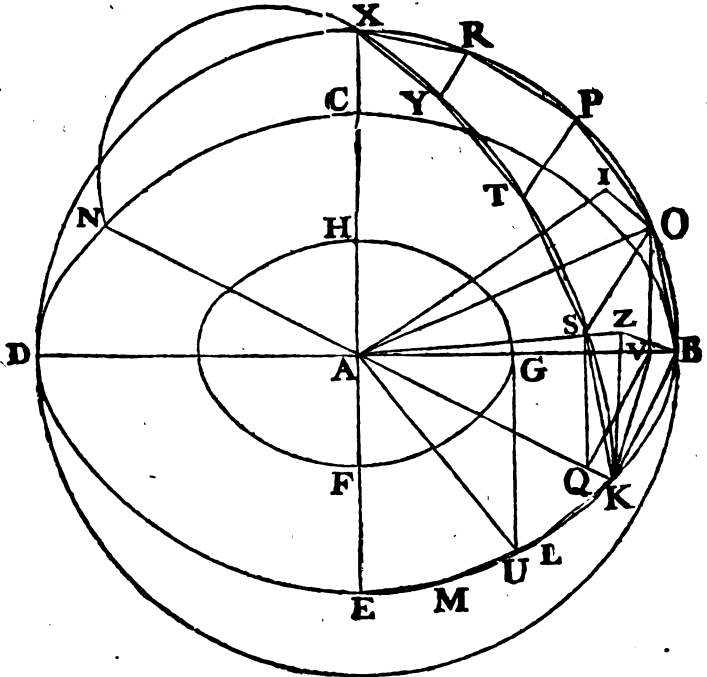
Therefore, if from the points, O, S, P, T, R, Y there be drawn
ftraight lines to the point A, there fhall be formed a folid po-
lyhedron between the circumferences BX, KX compofed of
pyramids the bafes of which are the quadrilaterals KBOS,
SOPT, TPRY, and the triangle YRX, and of which the com-
mon vertex is the point A: And if the fame conftruction be
made upon each of the fides KL, LM, ME, as has been done
upon BK, and the like be done alfo in the other three qua-
drants, and in the other hemifphere; there fhall be for-
med a folid polyhedron defcribed in the fphere, compo-
                                                    fed

fed of pyramids, the bafes of which are the aforefaid qua-
drilateral figures, and the triangle YRX, and thofe formed in
the like manner in the reft of the fphere, the common vertex
of them all being the point A: And the fuperficies of this fo-
lid polyhedron does not meet the leffer fphere in which is the
circle FGH: For, from the point A draw [a] AZ perpendicular
to the plane of the quadrilateral KBOS meeting it in Z, and
join BZ, ZK: And becaufe AZ is perpendicular to the plane
KBOS, it makes right angles with every ftraight line meeting
it in that plane; therefore AZ is perpendicular to BZ and ZK:
And becaufe AB is equal to AK, and that the fquares of of AZ,
ZB, are equal to the fquare of AB; and the fquares of AZ,
ZK to the fquare of AK [b]; therefore the fquares of AZ, ZB
are equal to the fquares of AZ, ZK: Take from thefe equals
the fquare of AZ, the remaining fquare of BZ is equal to the
remaining fquare of ZK; and therefore the ftraight line BZ
is equal to ZK: In the like manner it may be demonftrated,
that the ftraight lines drawn from the point Z to the points O,
S are equal to BZ or ZK: Therefore the circle defcribed from
the centre Z, and diftance ZB fhall pafs through the points K, O,
S, and KBOS fhall be a quadrilateral figure in the circle: And
becaufe KB is greater than QV, and QV equal to SO, there-
fore KB is greater than SO: But KB is equal to each of the
ftraight lines BO, KS; wherefore each of the circumferences
cut off by KB, BO, KS is greater than that cut off by OS; and
thefe three circumferences, together with a fourth equal to one
of them, are greater than the fame three together with that cut
off by OS; that is, than the whole circumference of the cir-
cle; therefore the circumference fubtended by KB is greater
than the fourth part of the whole circumference of the circle
KBOS, and confequently the angle BZK at the centre is great-
er than a right angle: And becaufe the angle BZK is obtufe,
the fquare of BK is greater [c] than the fquares of BZ, ZK;
that is, greater than twice the fquare of BZ. Join KV, and
becaufe in the triangles KBV, OBV, KB, BV are equal to OB,
BV, and that they contain equal angles; the angle KVB is e-
qual [d] to the angle OVB: And OVB is a right angle; there-
fore alfo KVB is a right angle: And becaufe BD is lefs than
twice DV, the rectangle contained by DB, BV is lefs than
twice the rectangle DVB; that is [e], the fquare of KB is lefs
than twice the fquare of KV: But the fquare of KB is greater
than twice the fquare of BZ; therefore the fquare of KV is
greater

T

greater than the square of BZ: And becaufe BA is equal to AK, and that the fquares of BZ, ZA are equal together to the fquare of BA, and the fquares of KV, VA to the fquare of AK; therefore the fquares of BZ, ZA are equal to the fquares of KV, VA; and of thefe the fquare of KV is greater than the fquare of BZ; therefore the fquare of VA is lefs than the fquare of ZA, and the ftraight line AZ greater than VA: Much more then is AZ greater than AG; becaufe, in the preceding propofition, it was fhown that KV falls without the circle FGH: And AZ is perpendicular to the plane KBOS, and is therefore the fhorteft of all the ftraight lines that can be drawn from A, the centre of the fphere to that plane. Therefore the plane KBOS does not meet the leffer fphere.

And that the other planes between the quadrants BX, KX fall without the leffer fphere, is thus demonftrated: From the point A draw AI perpendicular to the plane of the quadrilateral SOPT, and join IO; and, as was demonftrated of the plane KBOS and the point Z, in the fame way it may be fhown that the point I is the centre of a circle defcribed about SOPT: and that OS is greater than PT; and PT was fhown to be parallel to OS: Therefore, becaufe the two trapeziums KBOS, SOPT infcribed in circles have their fides BK, OS parallel, as alfo OS, PT; and their other fides BO, KS, OP, ST all equal to one another, and that BK is greater than OS, and OS greater than PT, therefore the ftraight line ZB is greater * than IO. Join AO which will be equal to AB; and becaufe AIO, AZB are right angles, the fquares of AI, IO are equal to the fquare of AO or of AB; that is, to the fquares of AZ, ZB; and the fquare of ZB is greater than the fquare of IO, therefore the fquare of AZ is lefs than the fquare of AI; and the ftraight line AZ lefs than the ftraight line AI: And it was proved that AZ is greater than AG; much more then is AI greater than AG: Therefore the plane SOPT falls wholly without the leffer fphere: In the fame manner it may be demonftrated that the plane TPRY falls without the fame fphere, as alfo the triangle YRX, viz. by the Cor. of 2d Lemma. And after the fame way it may be demonftrated that all the planes which contain the folid polyhedron, fall without the leffer fphere. Therefore in the greater of two fpheres which have the fame centre, a folid polyhedron is defcribed, the fuperficies of which does not meet the leffer fphere. Which was to be done.

a 2. Lem. 12.

But

But the ſtraight line AZ may be demonſtrated to be greater than AG otherwiſe, and in a ſhorter manner, without the help of Prop. 16. as follows. From the point G draw GU at right angles to AG and join AU. If then the circumference BE be biſected, and its half again biſected, and ſo on, there will at length be left a circumference leſs than the circumference which is ſubtended by a ſtraight line equal to GU inſcribed in the circle BCDE : Let this be the circumference KB : Therefore the ſtraight line KB is leſs than GU : And becauſe the angle BZK is obtuſe, as was proved in the preceding, therefore BK is greater than BZ : But GU is greater than BK ; much more then is GU greater than BZ, and the ſquare of GU than the ſquare of BZ ; and AU is equal to AB ; therefore the ſquare of AU, that is, the ſquares of AG, GU are equal to the ſquare of AB, that is, to the ſquares of AZ, ZB ; but the ſquare of BZ is leſs than the ſquare of GU ; therefore the ſquare of AZ is greater than the ſquare of AG, and the ſtraight line AZ conſequently greater than the ſtraight line AG.

Cor. And if in the leſſer ſphere there be deſcribed a ſolid polyhedron by drawing ſtraight lines betwixt the points in which the ſtraight lines from the centre of the ſphere drawn to all the angles of the ſolid polyhedron in the greater ſphere meet the ſuperficies of the leſſer ; in the ſame order in which are joined the points in which the ſame lines from the centre meet the ſuperficies of the greater ſphere ; the ſolid polyhedron in the ſphere BCDE has to this other ſolid polyhedron the triplicate ratio of that which the diameter of the ſphere BCDE has to the diameter of the other ſphere : For if theſe two ſolids be divided into the ſame number of pyramids, and in the ſame order ; the pyramids ſhall be ſimilar to one another, each to each : Becauſe they have the ſolid angles at their common vertex, the centre of the ſphere, the ſame in each pyramid, and their other ſolid angle at the baſes equal to one another, each to each [a], becauſe they are contained by three plane angles equal each to each ; and the pyramids are contained by the ſame number of ſimilar planes ; and are therefore ſimilar [b] to one another, each to each : But ſimilar pyramids have to one another the triplicate [c] ratio of their homologous ſides. Therefore the pyramid of which the baſe is the quadrilateral KBOS, and vertex A, has to the pyramid in the other ſphere of the ſame order, the triplicate ratio of their homologous

a B. 11.

b 11. def. 11.

c Cor. 12.

T 2   ſides ;

fides ; that is, of that ratio which AB from the centre of the greater fphere has to the ftraight line from the fame centre to the fuperficies of the leſſer fphere. And in like manner, each pyramid in the greater fphere has to each of the fame order in the leſſer, the triplicate ratio of that which AB has to the femidiameter of the leſſer fphere. And as one antecedent is to its confequent, fo are all the antecedents to all the confequents. Wherefore the whole ſolid polyhedron in the greater fphere has to the whole folid polyhedron in the other, the triplicate ratio of that which AB the femidiameter of the firſt has to the femidiameter of the other ; that is, which the diameter BD of the greater has to the diameter of the other fphere.

PROP. XVIII. THEOR.

SPHERES have to one another the triplicate ratio of that which their diameters have.

Let ABC, DEF be two fpheres of which the diameters are BC, EF. The fphere ABC has to the fphere DEF the triplicate ratio of that which BC has to EF.

For, if it has not, the fphere ABC ſhall have to a fphere either leſs or greater than DEF, the triplicate ratio of that which BC has to EF. Firſt, let it have that ratio to a leſs, viz. to the fphere GHK ; and let the fphere DEF have the fame centre with GHK ; and in the greater fphere DEF defcribe [a]
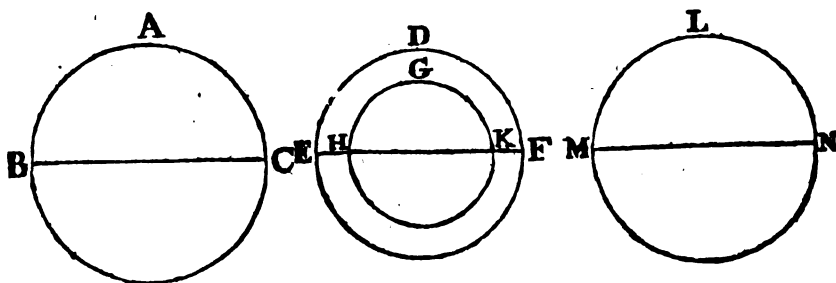
a ſolid polyhedron, the fuperficies of which does not meet the leſſer fphere GHK ; and in the fphere ABC defcribe another fimilar to that in the fphere DEF : Therefore the folid polyhedron in the fphere ABC has to the folid polyhedron in the fphere DEF, the triplicate ratio [b] of that which BC has to EF. But the fphere ABC has to the fphere GHK, the triplicate ratio

tio

tio of that which BC has to EF ; therefore, as the sphere ABC
to the sphere GHK, so is the solid polyhedron in the sphere ABC
to the solid polyhedron in the sphere DEF : But the sphere
ABC is greater than the solid polyhedron in it ; therefore ᵉ al- c 14. 5.
so the sphere GHK is greater than the solid polyhedron in the
sphere DEF : But it is also less, because it is contained within
it, which is impossible : Therefore the sphere ABC has not to
any sphere less than DEF, the triplicate ratio of that which
BC has to EF. In the same manner, it may be demonstrated,
that the sphere DEF has not to any sphere less than ABC, the
triplicate ratio of that which EF has to BC. Nor can the
sphere ABC have to any sphere greater than DEF, the tripli-
cate ratio of that which BC has to EF : For, if it can, let it
have that ratio to a greater sphere LMN : Therefore, by inver-
sion, the sphere LMN has to the sphere ABC, the triplicate
ratio of that which the diameter EF has to the diameter BC.
But, as the sphere LMN to ABC, so is the sphere DEF to some
sphere, which must be less ᵉ than the sphere ABC, because the
sphere LMN is greater than the sphere DEF : Therefore the
sphere DEF has to a sphere less than ABC the triplicate ratio
of that which EF has to BC ; which was shewn to be impos-
sible : Therefore the sphere ABC has not to any sphere greater
than DEF the triplicate ratio of that which BC has to EF : And
it was demonstrated, that neither has it that ratio to any sphere
less than DEF. Therefore the sphere ABC has to the sphere
DEF, the triplicate ratio of that which BC has to EF. Q. E. D.

F I N I S.

T 3

N O T E S

CRITICAL AND GEOMETRICAL;

CONTAINING

An account of those Things in which this Edition differs
from the Greek Text; and the Reasons of the Alte-
rations which have been made. As also Observations
on some of the Propositions.

By R O B E R T   S I M S O N, M. D.

Emeritus Professor of Mathematics in the University of Glasgow.

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M,DCC,LXXXI.

N O T E S, &c.

DEFINITION I. BOOK I.

IT is neceffary to confider a folid, that is, a magnitude which
has length, breadth, and thicknefs, in order to underftand
aright the definitions of a point, line, and fuperficies ; for thefe
all arife from a folid, and exift in it : The boundary, or boun-
daries which contain a folid are called fuperficies, or the boun-
dary which is common to two folids which are contiguous, or
which divides one folid into two contiguous parts, is called a
fuperficies : Thus, if BCGF be one of the boundaries which
contain the folid ABCDEFGH, or which is the common
boundary of this folid, and the folid BKLCFNMG, and is there-
fore in the one as well as the other folid, is called a fuperficies,
and has no thicknefs: For if it have any, this thicknefs' muft
either be a part of the thicknefs
of the folid AG, or the folid
BM, or a part of the thicknefs of
each of them. It cannot be a
part of the thicknefs of the folid
BM ; becaufe, if this folid be re-
moved from the folid AG, the
fuperficies BCGF, the boundary
of the folid AG, remains ftill the
fame as it was. Nor can it be a
part of the thicknefs of the folid AG ; becaufe, if this be re-
moved from the folid BM, the fuperficies BCGF, the boundary
of the folid BM, does neverthelefs remain ; therefore the fuper-
ficies BCGF has no thicknefs, but only length and breadth.
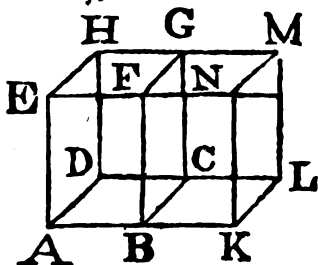
The boundary of a fuperficies is called a line, or a line is the
common boundary of two fuperficies that are contiguous, or
which divides one fuperficies into two contiguous parts: Thus,
if BC be one of the boundaries which contain the fuperficies
ABCD, or which is the common boundary of this fuperficies,
and of the fuperficies KBCL which is contiguous to it, this
boundary BC is called a line, and has no breadth: For, if it
have any, this muft be part either of the breadth of the fuper-
ficies ABCD, or of the fuperficies KBCL, or part of each of
them. It is not part of the breadth of the fuperficies KBCL ;
for, if this fuperficies be removed from the fuperficies ABCD,

the

the line BC which is the boundary of the superficies ABCD re-
mains the same as it was : Nor can the breadth that BC is sup-
pofed to have, be a part of the breadth of the superficies ABCD ;
becaufe, if this be removed from the superficies KBCL, the line
BC which is the boundary of the superficies KBCL does never-
thelefs remain : Therefore the line BC has no breadth : And be-
caufe the line BC is in a superficies, and that a superficies has
no thicknefs, as was fhewn ; therefore a line has neither breadth
nor thicknefs, but only length.

The boundary of a line is called a point, or a point is the
common boundary or extremity
of two lines that are contiguous :
Thus, if B be the extremity of the
line AB, or the common extre-
mity of the two lines AB, KB,
this extremity is called a point,
and has no length : For, if it have
any, this length muft either be
part of the length of the line AB,
or of the line KB· It is not part
of the length of KB ; for, if the line KB be removed from AB,
the point B which is the extremity of the line AB remains the
fame as it was : Nor is it part of the length of the line AB ;
for, if AB be removed from the line KB, the point B, which is
the extremity of the line KB, does neverthelefs remain : There-
fore the point B has no length : And becaufe a point is in a line,
and a line has neither breadth nor thicknefs, therefore a point
has no length, breadth, nor thicknefs.   And in this manner the
definitions of a point, line, and superficies are to be underftood.
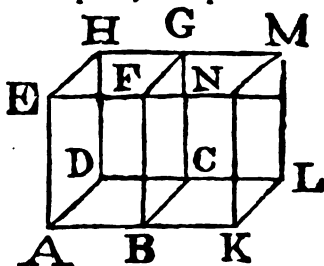
D E F. VII. B. I.

Inftead of this definition as it is in the Greek copies, a more
diftinct one is given from a property of a plane superficies, which
is manifeftly fuppofed in the elements, viz. that a ftraight line
drawn from any point in a plane to any other in it, is wholly in
that plane.

D E F. VIII. B. I.

It feems that he who made this definition defigned that it
fhould comprehend not only a plane angle contained by two
ftraight lines, but likewife the angle which fome conceive to
be made by a ftraight line and a curve, or by two curve lines,
which meet one another in a plane : But, tho' the meaning of
the

the words ᴏɴ' ɪᴏᴛᴜᴀɪ, that is, in a ſtraight line, or in the ſame direction, be plain, when two ſtraight lines are ſaid to be in a ſtraight line, it does not appear what ought to be underſtood by theſe words, when a ſtraight line and a curve, or two curve lines, are ſaid to be in the ſame direction; at leaſt it cannot be explained in this place; which makes it probable that this definition, and that of the angle of a ſegment, and what is ſaid of the angle of a ſemicircle, and the angles of ſegments, in the 16. and 31. propoſitions of book 3. are the additions of ſome leſs ſkilful editor: On which account, eſpecially ſince they are quite uſeleſs, theſe definitions are diſtinguiſhed from the reſt by inverted double commas.

DEF. XVII. B. I.

The words, " which alſo divides the circle into two equal " parts," are added at the end of this definition in all the copies, but are now left out as not belonging to the definition, being only a corollary from it. Proclus demonſtrates it by conceiving one of the parts into which the diameter divides the circle, to be applied to the other; for it is plain they muſt coincide, elſe the ſtraight lines from the centre to the circumference would not be all equal: The ſame thing is eaſily deduced from the 31. prop. of book 3. and the 24 of the ſame; from the firſt of which it follows that ſemicircles are ſimilar ſegments of a circle: And from the other, that they are equal to one another.

DEF. XXXIII. B. I.

This definition has one condition more than is neceſſary; becauſe every quadrilateral figure which has its oppoſite ſides equal to one another, has likewiſe its oppoſite angles equal; and on the contrary.

Let ABCD be a quadrilateral figure of which the oppoſite ſides AB, CD are equal to one another; as alſo AD and BC: Join BD; the two ſides AD, DB are equal to the two CB, BD, and the baſe AB is equal to the baſe CD; therefore by prop. 8. of book 1. the angle ADB is equal to the angle CBD; and by prop. 4. B. 1. the angle BAD is equal to the angle DCB, and ABD to BDC; and therefore alſo the angle ADC is equal to the angle ABC.

And
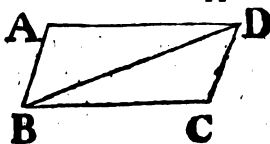
And if the angle BAD be equal to the oppofite angle BCD, and the angle ABC to ADC; the oppofite fides are equal: Becaufe, by prop. 32. B. 1. all the angles of the quadrilateral figure ABCD are together equal to four right angles, and the two angles BAD, ADC are together equal to the two angles BCD, ABC: Wherefore BAD, ADC are the half of all the four angles; that is, BAD and ADC are equal to two right angles: And therefore AB, CD are parallels by prop. 28. B. 1. In the fame manner AD, BC are parallels: Therefore ABCD is a parallelogram, and its oppofite fides are equal by 34. prop. B. 1.
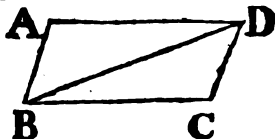
P R O P. VII. B. I.

There are two cafes of this propofition, one of which is not in the Greek text, but is as neceffary as the other: And that the cafe left out has been formerly in the text appears plainly from this, that the fecond part of prop. 5. which is neceffary to the demonftration of this cafe, can be of no ufe at all in the elements, or any where elfe, but in this demonftration; becaufe the fecond part of prop. 5. clearly follows from the firft part, and prop. 13. B. 1. This part muft therefore have been added to prop. 5. upon account of fome propofition betwixt the 5. and 13. but none of thefe ftand in need of it except the 7. propofition, on account of which it has been added: Befides, the tranflation from the Arabic has this cafe explicitly demonftrated: And Proclus acknowledges that the fecond part of prop. 5. was added upon account of prop. 7. but gives a ridiculous reafon for it, " that it might afford an anfwer to objections made " againft the 7." as if the cafe of the 7. which is left out, were, as he exprefsly makes it, an objection againft the propofition itfelf. Whoever is curious may read what Proclus fays of this in his commentary on the 5. and 7. propofitions; for it is not worth while to relate his trifles at full length.

It was thought proper to change the enunciation of this 7. prop. fo as to preferve the very fame meaning; the literal tranflation from the Greek being extremely harfh, and difficult to be underftood by beginners.

P R O P.

PROP. XI. B. I.

A corollary is added to this propofition, which is neceffary to Prop. 1. b. 11. and otherwife.

PROP. XX. and XXI. B. I.

Proclus, in his commentary, relates, that the Epicureans derided this propofition, as being manifeft even to affes, and needing no demonstration; and his anfwer is, that though the truth of it be manifeft to our fenfes, yet it is fcience which muft give the reafon why two fides of a triangle are greater than the third: But the right anfwer to this objection againft this and the 21ft, and fome other plain propofitions, is, that the number of axioms ought not to be increafed without neceffity, as it muft be if thefe propofitions be not demonftrated. Monf. Clairault, in the preface to his elements of geometry, publifhed in French at Paris, ann. 1741, fays, That Euclid has been at the pains to prove, that the two fides of a triangle which is included within another are together lefs than the two fides of the triangle which includes it; but he has forgot to add this condition, viz. that the triangles muft be upon the fame bafe; becaufe, unlefs this be added, the fides of the included triangle may be greater than the fides of the triangle which includes it, in any ratio which is lefs than that of two to one, as Pappus Alexandrinus has demonftrated in Prop. 3. b. 3. of his mathematical collections.

PROP. XXII. B. I.

Some authors blame Euclid becaufe he does not demonftrate, that the two circles made ufe of in the conftruction of this problem muft cut one another: But this is very plain from the determination he has given, viz. that any two of the ftraight lines DF, FG, GH muft be greater than the third: For who is fo dull, tho' only beginning to learn the elements, as not to perceive that the circle defcribed from the centre F, at the diftance FD, muft meet FH betwixt F and H, becaufe FD is lefs than FH; and that, for the like reafon, the circle defcribed from the centre G, at the diftance GH or GM, muft meet DG betwixt D

and
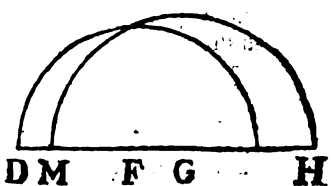
and G ; and that thefe circles muſt meet one another, becauſe
FD and GH are together greater
than FG ?   And this determina-
tion is eaſier to be underſtood than
that which Mr Thomas Simpſon
derives from it, and puts inſtead
of Euclid's, in the 49th page of
his elements of geometry, that he
may ſupply the omiſſion he blames

D M   F G   H

Euclid for ; which determination is, that any of the three
ſtraight lines muſt be leſs than the ſum, but greater than the
difference of the other two : From this he ſhews the circles muſt
meet one another, in one caſe ; and ſays, that it may be proved
after the fame manner in any other caſe :~ But the ſtraight line
GM which he bids take from GF may be greater than it, as in
the figure here annexed ; in which caſe his demonſtration muſt
be changed into another.

P R O P. XXIV.   B. I.

To this is added, " of the two ſides DE, DF, let DE be
" that which is not greater than the other ;" that is, take that
ſide of the two DE, DF which is not greater than the other, in
order to make with it the angle EDG
equal to BAC ; becauſe, without this
reſtriction, there might be three differ-
ent caſes of the propoſition, as Campa-
nus and others make.

Mr Thomas Simpſon, in p. 262. of
the ſecond edition of his elements of
geometry, printed anno 1760, obſerves,
in his notes, that it ought to have been
ſhown, that the point F falls below the
line EG ; this probably Euclid omitted,

D

E

G

F

, as it is very eaſy to perceive, that DG being equal to DF, the
point G is in the circumference of a circle deſcribed from the
centre D at the diſtance DF, and muſt be in that part of it
which is above the ſtraight line EF, becauſe DG falls above
DF, the angle EDG being greater than the angle EDF.
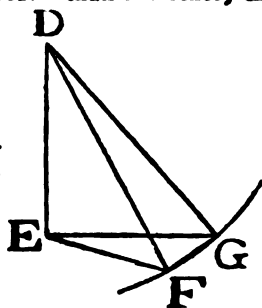
P R O P. XXIX.   B. I.

The propoſition which is uſually called the 5th poſtulate, or
11th axiom, by ſome the 12th, on which this 29th depends, has
<div align="right">given</div>

given a great deal to do, both to antient and modern geome-
ters : It feems not to be properly placed among the Axioms,
as, indeed, it is not felf-evident; but it may be demonftrated
thus :

DEFINITION 1.

The diftance of a point from a ftraight line, is the perpendi-
cular drawn to it from the point.

DEF. 2.

One ftraight line is faid to go nearer to, or further from, ano-
ther ftraight line, when the diftances of the points of the firft
from the other ftraight line become lefs or greater than they
were ; and two ftraight lines are faid to keep the fame diftance
from one another, when the diftance of the points of one of them
from the other is always the fame.

AXIOM.

A ftraight line cannot firft come nearer to one another ftraight
line, and then go further from
it, before it cuts it ; and, in like
manner, a ftraight line cannot
go further from another ftraight
line, and then come nearer to
it ; nor can a ftraight line keep
the fame diftance from another ftraight line, and then come
nearer to it, or go further from it ; for a ftraight line keeps al-
ways the fame direction.
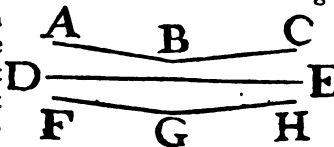
For example, the ftraight line ABC cannot firft come near-
er to the ftraight line DE, as
from the point A to the point
B, and then, from the point B
to the point C, go further from
the fame DE : And, in like man-
ner, the ftraight line FGH can-
not go further from DE, as from F to G, and then, from G to
H, come nearer to the fame DE : And fo in the laft cafe, as in
fig. 2.
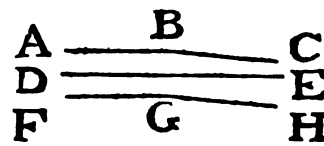
See the fi-
gure above.

PROP. I.

If two equal ftraight lines AC, BD, be each at right angles
to the fame ftraight line AB ; if the points C, D be joined by
the ftraight line CD, the ftraight line EF drawn from any point
E in AB unto CD, at right angles to AB, fhall be equal to AC,
or BD.

If EF be not equal to AC, one of them muft be greater
than the other ; let AC be the greater ; then, becaufe FE is
lefs

Book I. lefs than CA, the ftraight line CFD is nearer to the ftraight
line AB at the point F than at the
point C, that is, CF comes nearer
to AB from the point C to F: But
becaufe DB is greater than FE,
the ftraight line CFD is further
from AB at the point D than at F,
that is, FD goes further from AB
from F to D: Therefore the ftraight
line CFD firft comes nearer to the
ftraight line AB, and then goes further from it, before it cuts
it; which is impoffible. If FE be faid to be greater than CA,
or DB, the ftraight line CFD firft goes further from the ftraight
line AB, and then comes nearer to it; which is alfo impoffible.
Therefore FE is not unequal to AC, that is, it is equal to it.
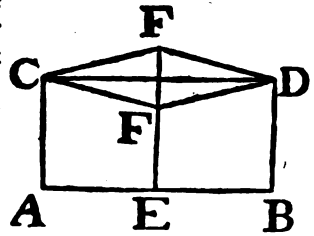
P R O P. 2.

If two equal ftraight lines AC, BD be each at right angles to
the fame ftraight line AB; the ftraight line CD which joins
their extremities make right angles with AC and BD.

Join AD, BC; and becaufe, in the triangles CAB, DBA,
CA, AB are equal to DB, BA, and the angle CAB equal to
a 4. 1.   the angle DBA; the bafe BC is equal ᵃ to the bafe AD: And
in the triangles ACD, BDC, AC, CD are equal to BD, DC,
and the bafe AD is equal to the bafe
BC: Therefore the angle ACD is e-
b 8. 1.   qual ᵇ to the angle BDC: From any
point E in AB draw EF unto CD,
at right angles to AB; therefore, by
Prop. 1. EF is equal to AC, or BD;
wherefore, as has been juft now
fhown, the angle ACF is equal to
the angle EFC: In the fame manner, the angle BDF is equal
to the angle EFD; but the angles ACD, BDC are equal;
therefore the angles EFC and EFD are equal, and right
c 10. def.   angles ᶜ; wherefore alfo the angles ACD, BDC are right
1.   angles.
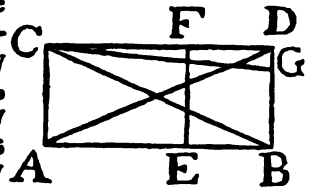
Cor. Hence, if two ftraight lines AB, CD be at right angles
to the fame ftraight line AC, and if betwixt them a ftraight
line BD be drawn at right angles to either of them, as to AB;
then BD is equal to AC, and BDC is a right angle.

If AC be not equal to BD, take BG equal to AC, and
join CG: Therefore, by this Propofition, the angle ACG is a
right angle; but ACD is alfo a right angle; wherefore the an-
gles

gles ACD, ACG are equal to one another, which is impoffible. **Book I.**
Therefore BD is equal to AC; and by this propofition BDC is
a right angle.

P R O P. 3.

If two ftraight lines which contain an angle be produced,
there may be found in either of them a point from which the
perpendicular drawn to the other fhall be greater than any given
ftraight line.

Let AB, AC be two ftraight lines which make an angle with
one another, and let AD be the given ftraight line; a point
may be found either in AB or AC, as in AC, from which the
perpendicular drawn to the other AB fhall be greater than AD.

In AC take any point E, and draw EF perpendicular to
AB; produce AE to G, fo that EG be equal to AE; and
produce FE to H, and make EH equal to FE, and join HG.
Becaufe, in the triangles AEF, GEH, AE, EF are equal to
GE, EH, each to each, and contain equal ᵃ angles, the angle **a 15. 1.**
GHE is therefore equal ᵇ to the angle AFE which is a right **b. 4. 1.**
angle: Draw GK perpendicular to AB; and becaufe the ftraight
lines FK, HG
are at right an-
gles to FH, and
KG at right an-
gles to FK; KG
is equal to FH,
by Cor. Pr. 2.
that is, to the
double of FE.
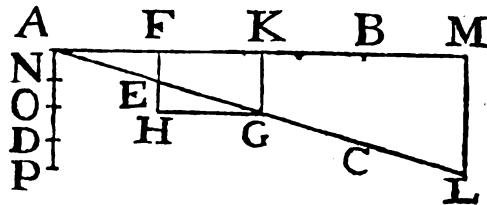
In the fame manner, if AG be produced to L, fo that GL be
equal to AG, and LM be drawn perpendicular to AB, then
LM is double of GK, and fo on. In AD take AN equal to
FE, and AO equal to KG, that is, to the double of FE, or
AN; alfo, take AP equal to LM, that is, to the double of KG,
or AO; and let this be done till the ftraight line taken be great-
er than AD: Let this ftraight line fo taken be AP, and becaufe
AP is equal to LM, therefore LM is greater than AD. Which
was to be done.

P R O P. 4.

If two ftraight lines AB, CD make equal angles EAB,
ECD with another ftraight line EAC towards the fame parts of
it; AB and CD are at right angles to fome ftraight line.

U   Bifect

Bisect AC in F, and draw FG perpendicular to AB; take CH in the straight line CD equal to AG, and on the contrary side of AC to that on which AG is, and join FH : Therefore, in the triangles AFG, CFH the sides FA, AG are equal to FC, CH, each to each, and the angle

a 15. 1.
b 4. 1.

FAG, that [a] is, EAB is equal to the angle FCH; wherefore [b] the angle AGF is equal to CHF, and AFG to the angle CFH : To these last add the common angle AFH ; therefore the two angles AFG, AFH are equal to the two angles CFH, HFA, which two last are equal together to two right angles [c], therefore also AFG,

c 13. 1.
d 14. 1.

AFH are equal to two right angles, and consequently [d] GF and FH are in one straight line.   And because AGF is a right angle, CHF which is equal to it is also a right angle : Therefore the straight lines AB, CD are at right angles to GH.
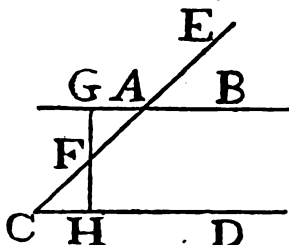
P R O P. 5.

If two straight lines AB, CD be cut by a third ACE so as to make the interior angles BAC, ACD, on the same side of it, together less than two right angles ; AB and CD being produced shall meet one another towards the parts on which are the two angles which are less than two right angles.

a 23. 1.

At the point C in the straight line CE make [a] the angle ECF equal to the angle EAB, and draw to AB the straight line CG at right angles to CF : Then, because the angles ECF, EAB are equal to one an-

b 13. 1.

other, and that the angles ECF, FCA are together e- qual [b] to two right angles, the angles EAB, FCA are equal to two right angles. But, by the hypothesis, the angles EAB, ACD are to- gether less than two right angles ; therefore the angle FCA is greater than ACD, and CD falls between CF and AB : And because CF and CD make an angle with one another, by Prop. 3. a point may be found in either of them CD from which the perpendicular drawn to CF shall be greater than the straight line CG : Let this
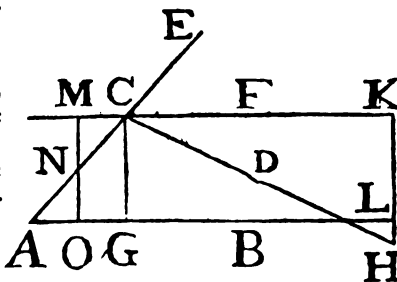
this point be H, and draw HK perpendicular to CF meeting
AB in L: And becaufe AB, CF contain equal angles with
AC on the fame fide of it, by Prop. 4. AB and CF are at right
angles to the ftraight line MNO which bifects AC in N and is
perpendicular to CF: Therefore, by Cor. Prop. 2. CG and KL
which are at right angles to CF are equal to one another: And
HK is greater than CG, and therefore is greater than KL, and
confequently the point H is in KL produced. Wherefore the
ftraight lines CDH drawn betwixt the points C, H which are on
contrary fides of AL, muft neceffarily cut the ftraight line AB.

P R O P. XXXV. B. I.

The demonftration of this Propofition is changed, becaufe, if
the method which is ufed in it was followed, there would be
three cafes to be feparately demonftrated, as is done in the
tranflation from the Arabic; for, in the Elements, no cafe of a
Propofition that requires a different demonftration, ought to
be omitted. On this account, we have chofen the method
which Monf. Clairault has given, the firft of any, as far as I
know, in his Elements, page 21. and which afterwards Mr
Simpfon gives in his page 32. But whereas Mr Simpfon makes
ufe of Prop. 26. B. 1. from which the equality of the two
triangles does not immediately follow, becaufe, to prove that,
the 4. of B. 1. muft likewife be made ufe of, as may be feen,
in the very fame cafe in the 34. Prop. B. 1. it was thought bet-
ter to make ufe only of the 4. of B. 1.

P R O P. XLV. B I.

The ftraight line KM is proved to be parallel to FL from
the 33. Prop.; whereas KH is parallel to FG by conftruction,
and KHM, FGL have been demonftrated to be ftraight lines.
A corollary is added from Commandine, as being often ufed.

P R O P. XIII. B. II.

IN this Propofition only acute angled triangles are mention-
ed, whereas it holds true of every triangle: And the de-
monftrations of the cafes omitted are added; Commandine and
Clavius have likewife given their demonftrations of thefe cafes.

P R O P. XIV. B. II.

In the demonftration of this, fome Greek editor has ig-
norantly inferted the words, " but if not, one of the two BE,

U 2   " ED

" ED is the greater ; let BE be the greater, and produce it to
" F," as if it was of any confequence whether the greater or
leffer be produced : Therefore, inftead of thefe words, there
ought to be read only, " but if not, produce BE to F."

P R O P. I.   B. III.

SEVERAL authors, efpecially among the modern mathe-
maticians and logicians, inveigh too feverely againft indi-
rect or Apagogic demonftrations, and fometimes ignorantly
enough ; not being aware that there are fome things that can-
not be demonftrated any other way : Of this the prefent pro-
pofition is a very clear inftance, as no direct demonftration
can be given of it : Becaufe, befides the definition of a circle,
there is no principle or property relating to a circle antecedent
to this problem, from which either a direct or indirect de-
monftration can be deduced : Wherefore it is neceffary that the
point found by the conftruction of the problem be proved to
be the centre of the circle, by the help of this definition, and
fome of the preceding propofitions : And becaufe, in the de-
monftration, this propofition muft be brought in, viz. ftraight
lines from the centre of a circle to the circumference are equal,
and that the point found by the conftruction cannot be affu-
med as the centre, for this is the thing to be demonftrated ; it
is manifeft fome other point muft be affumed as the centre ;
and if from this affumption an abfurdity follows, as Euclid de-
monftrates there muft, then it is not true that the point affumed
is the centre ; and as any point whatever was affumed, it follows
that no point, except that found by the conftruction, can be the
centre; from which the neceffity of an indirect demonftration in
this cafe is evident.

P R O P. XIII.   B. III.

As it is much eafier to imagine that two circles may touch
one another within in more points than one, upon the fame
fide, than upon oppofite fides ; the figure of that cafe ought
not to have been omitted ; but the conftruction in the Greek
text would not have fuited with this figure fo well, becaufe the
centres of the circles muft have been placed near to the cir-
cumferences : On which account another conftruction and de-
monftration is given, which is the fame with the fecond part
of that which Campanus has tranflated from the Arabic,
where

NOTES.

where without any reason the demonstration is divided into two parts.

P R O P. XV. B. III.

The converse of the second part of this proposition is wanting, though in the preceding, the converse is added, in a like case, both in the enunciation and demonstration; and it is now added in this. Besides, in the demonstration of the first part of this 15th, the diameter AD (see Commandine's figure) is proved to be greater than the straight line BC by means of another straight line MN; whereas it may be better done without it: On which accounts we have given a different demonstration, like to that which Euclid gives in the preceding 14th, and to that which Theodosius gives in prop. 6. B. 1. of his Spherics, in this very affair.

P R O P. XVI. B. III.

In this we have not followed the Greek nor the Latin translation literally, but have given what is plainly the meaning of the proposition, without mentioning the angle of the semicircle, or that which some call the cornicular angle which they conceive to be made by the circumference and the straight line which is at right angles to the diameter, at its extremity; which angles have furnished matter of great debate between some of the modern geometers, and given occasion of deducing strange consequences from them, which are quite avoided by the manner in which we have expressed the proposition. And in like manner, we have given the true meaning of prop. 31. b. 3. without mentioning the angles of the greater or lesser segments: These passages, Vieta, with good reason, suspects to be adulterated, in the 386th page of his Oper. Math.

P R O P. XX. B. III.

The first words of the second part of this demonstration, " κεκλασθω δη παλιν," are wrong translated by Mr Briggs and Dr Gregory " Rursus inclinetur;" for the translation ought to be " Rursus inflectatur," as Commandine has it: A straight line is said to be inflected either to a straight, or curve line, when a straight line is drawn to this line from a point, and from the point in which it meets it, a straight line making an angle with the former is drawn to another point, as is evident from the 90th prop. of Euclid's Data: For thus the whole line betwixt the first and last points, is inflected or broken at

the point of inflection, where the two straight lines meet. And in the like sense two straight lines are said to be inflected from two points to a third point, when they make an angle at this point; as may be seen in the description given by Pappus Alexandrinus of Apollonius's Books de Locis planis, in the preface to his 7th book: We have made the expression fuller from the 90th prop. of the data.

P R O P. XXI.   B. III.

There are two cases of this proposition, the second of which, viz. when the angles are in a segment not greater than a semicircle, is wanting in the Greek: And of this a more simple demonstration is given than that which is in Commandine, as being derived only from the first case, without the help of triangles.

P R O P. XXIII. and XXIV.   B. III.

In proposition 24. it is demonstrated, that the segment AEB must coincide with the segment CFD, (see Commandine's figure), and that it cannot fall otherwise, as CGD, so as to cut the other circle in a third point G, from this, that, if it did, a circle could cut another in more points than two: But this ought to have been proved to be impossible in the 23d prop. as well as that one of the segments cannot fall within the other: This part then is left out in the 24th, and put in its proper place, the 23d Proposition.

P R O P. XXV.   B. III.

This proposition is divided into three cases, of which two have the same construction and demonstration; therefore it is now divided only into two cases.

P R O P. XXXIII.   B. III.

This also in the Greek is divided into three cases, of which two, viz. one, in which the given angle is acute, and the other in which it is obtuse, have exactly the same construction and demonstration; on which account, the demonstration of the last case is left out as quite superfluous, and the addition of some unskilful editor; besides the demonstration of the case when the angle given is a right angle, is done a round about way, and is therefore changed to a more simple one, as was done by Clavius.

P R O P.

PROP. XXXV. B. III.

As the 25th and 33d propofitions are divided into more cafes, fo this 35th is divided into fewer cafes than are neceffary. Nor can it be fuppofed that Euclid omitted them becaufe they are eafy; as he has given the cafe, which by far, is the eafieft of them all, viz. that in which both the ftraight lines pafs through the centre: And in the following propofition he feparately demonftrates the cafe in which the ftraight line paffes through the centre, and that in which it does not pafs through the centre: So that it feems Theon, or fome other, has thought them too long to infert: But cafes that require different demonftrations, fhould not be left out in the elements, as was before taken notice of: Thefe cafes are in the tranflation from the Arabic, and are now put into the text

PROP. XXXVII. B. III.

At the end of this, the words, " in the fame manner it may " be demonftrated, if the centre be in AC," are left out as the addition of fome ignorant editor.

DEFINITIONS of BOOK IV.

WHEN a point is in a ftraight, or any other line, this point is by the Greek geometers faid απτισθαι, to be upon, or in that line, and when a ftraight line or circle meets a circle any way, the one is faid απτισθαι to meet the other: But when a ftraight line or circle meets a circle fo as not to cut it, it is faid εφαπτισθαι, to touch the circle; and thefe two terms are never promifcuoufly ufed by them: Therefore, in the 5th definition of B. 4. the compound εφαπτηται muft be read, inftead of the fimple απτηται: And in the 1ft, 2d, 3d, and 6th definitions in Commandine's tranflation, " tangit," muft be read inftead of " contingit:" And in the 2d and 3d definitions of Book 3. the fame change muft be made: But in the Greek text of propofitions 11th, 12th, 13th, 18th, 19th, Book 3. the compound verb is to be put for the fimple.

PROP. IV. B. IV.

In this, as alfo in the 8th and 13th propofitions of this book, it is demonftrated indirectly, that the circle touches a ftraight line; whereas in the 17th, 33d, and 37th propofitions of book 3. the fame thing is directly demonftrated: And this way we

have

have chofen to ufe in the propofitions of this book, as it is fhorter.

PROP. V. B. IV.

The demonftration of this has been fpoiled by fome unfkilful hand : For he does not demonftrate, as is neceffary, that the two ftraight lines which bifeƈt the fides of the triangle at right angles, muft meet one another ; and, without any reafon, he divides the propofition into three cafes ; whereas, one and the fame conftruƈtion and demonftration ferves for them all, as Campanus has obferved ; which ufelefs repetitions are now left out : The Greek text alfo in the Corollary is manifeftly vitiated, where mention is made of a given angle, though there neither is, nor can be any thing in the propofition relating to a given angle.

PROP. XV. and XVI. B. IV.

In the corollary of the firft of thefe, the words equilateral and equiangular are wanting in the Greek : And in prop. 16. inftead of the circle ABCD, ought to be read the circumference ABCD : Where mention is made of its containing fifteen equal parts.

DEF. III. B. V.

MANY of the modern mathematicians rejeƈt this definition : The very learned Dr Barrow has explained it at large at the end of his third leƈture of the year 1666, in which alfo he anfwers the objeƈtions made againft it as well as the fubjeƈt would allow : And at the end gives his opinion upon the whole, as follows :

" I fhall only add, that the author had, perhaps, no o-
" ther defign in making this definition, than (that he might
" more fully explain and embellifh his fubjeƈt) to give a gene-
" ral and fummary idea of ratio to beginners, by premifing
" this metaphyfical definition, to the more accurate defini-
" tions of ratios that are the fame to one another, or one of
" which is greater, or lefs than the other : I call it a meta-
" phyfical, for it is not properly a mathematical definition,
" fince nothing in mathematics depends on it, or is deduced,
" nor, as I judge, can be deduced from it : And the defini-
" tion of analogy, which follows, viz. Analogy is the fimi-
" litude

" litude of ratios, is of the fame kind, and can ferve for no
" purpofe in mathematics, but only to give beginners fome
" general, tho' grofs and confufed notion of analogy: But the
" whole of the doctrine of ratios, and the whole of mathema-
" tics, depend upon the accurate mathematical definitions which
" follow this : To thefe we ought principally to attend, as the
" doctrine of ratios is more perfectly explained by them ; this
" third, and others like it, may be entirely fpared without any
" lofs to geometry ; as we fee in the 7th book of the elements,
" where the proportion of numbers to one another is defined,
" and treated of, yet without giving any definition of the ratio
" of numbers ; tho' fuch a definition was as neceffary and ufe-
" ful to be given in that book, as in this: But indeed there is
" fcarce any need of it in either of them : Though I think that
" a thing of fo general and abftracted a nature, and thereby the
" more difficult to be conceived and explained, cannot be more
" commodioufly defined than as the author has done : Upon
" which account I thought fit to explain it at large, and defend
" it againft the captious objections of thofe who attack it." To
this citation from Dr Barrow I have nothing to add, except that
I fully believe the 3d and 8th definitions are not Euclid's, but
added by fome unfkilful editor.

D E F. XI. B. V.

It was neceffary to add the word "continual" before " pro-
" portionals" in this definition ; and thus it is cited in the 33d
prop. of book 11.

After this definition ought to have followed the definition of
compound ratio, as this was the proper place for it ; duplicate
and triplicate ratio being fpecies of compound ratio. But Theon
has made it the 5th def. of B. 6. where he gives an abfurd and
entirely ufelefs definition of compound ratio: For this reafon we
have placed another definition of it betwixt the 11th and 12th of
this book, which, no doubt, Euclid gave ; for he cites it ex-
prefsly in prop. 23. B. 6. and which Clavius, Herigon, and
Barrow, have likewife given, but they retain alfo Theon's, which
they ought to have left out of the elements.

D E F. XIII. B. V.

This, and the reft of the definitions following, contain the ex-
plication of fome terms which are ufed in the 5th and following
books ; which, except a few, are eafily enough underftood from
the

the propofitions of this book where they are firſt mentioned : They ſeem to have been added by Theon, or ſome other. However it be, they are explained ſomething more diſtinctly for the ſake of learners.

P R O P. IV. B. V.

In the conſtruction preceding the demonſtration of this, the words ά ιτυχι, any whatever, are twice wanting in ·the Greek, as alfo in the Latin tranſlations ; and are now added, as being wholly neceſſary.

Ibid. in the demonſtration ; in the Greek, and in the Latin tranſlation of Commandine, and in that of Mr Henry Briggs, which was publiſhed at London in 1620, together with the Greek text of the firſt ſix books, which tranſlation in this place is followed by Dr Gregory in his edition of Euclid, there is this ſentence following, viz. " and of A and C have been taken e-" quimultiples K, L ; and of B and D, any equimultiples " whatever (ά ιτυχι) M, N ;" which is not true, the words " any whatever," ought to be left out : And it is ſtrange that neither Mr Briggs, who did right to leave out theſe words in one place of prop. 13. of this book, nor Dr Gregory, who changed them into the word " ſome" in three places, and left them out in a fourth of that ſame prop. 13. did not alfo leave them out in this place of prop. 4. and in the ſecond of the two places where they occur in prop. 17. of this book, in neither of which they can ſtand confiſtent with truth : And in none of all theſe places, even in thoſe which they corrected in their Latin tranſlation, have they cancelled the words ά ιτυχι in the Greek text, as they ought to have done.

The ſame words ά ιτυχι are found in four places of prop. 11. of this book, in the firſt and laſt of which they are neceſſary, but in the ſecond and third, though they are true, they are quite ſuperfluous ; as they likewiſe are in the ſecond of the two places in which they are found in the 12th prop. and in the like places of prop. 22. 23. of this book ; but are wanting in the laſt place of prop. 23. as alſo in prop. 25. Book 11.

C O R. IV. P R O P. B. V.

This corollary has been unſkilfully annexed to this propoſition, and has been made inſtead of the legitimate demonſtration, which, without doubt, Theon, or ſome other editor, has taken away, not from this, but from its proper place in

this

this book : The author of it defigned to demonftrate, that if four Book V.
magnitudes E, G, F, H be proportionals, they are alfo propor-
tionals inverfely ; that is, G is to E, as H to F ; which is true ;
but the demonftration of it does not in the leaft depend upon
this 4th prop. or its demonftration : For, when he fays, " be-
" caufe it is demonftrated that if K be greater than M, L is
" greater than N," &c. This indeed is fhewn in the demon-
ftration of the 4th prop. but not from this, that E, G, F, H are
proportionals ; for this laft is the conclufion of the propofition.
Wherefore thefe words, " becaufe it is demonftrated," &c. are
wholly foreign to his defign : And he fhould have proved, that
if K be greater than M, L is greater than N, from this, that
E, G, F, H are proportionals, and from the 5th def. of this
book, which he has not ; but is done in propofition B, which we
have given in its proper place, inftead of this corollary ; and
another corollary is placed after the 4th prop. which is often of
ufe ; and is neceffary to the demonftration of prop. 18. of this
book.

P R O P. V. B. V.

In the conftruction which precedes the demonftration of
this propofition, it is required that EB may be the fame mul-
tiple of CG, that AE is of CF ; that is, that EB be divided
into as many equal parts, as there are parts in AE equal to
CF : From which it is evident, that this conftruction is not
Euclid's ; for he does not fhow the way of dividing ftraight
lines, and far lefs other magnitudes, into any number of equal
parts, until the 9th propofition of B 6. ; and he never requires
any thing to be done in the conftruction, of which he had not
before given the method of doing : For this rea-
fon, we have changed the conftruction to one,
which, without doubt, is Euclid's, in which no-
thing is required but to add a magnitude to itfelf a
certain number of times ; and this is to be found
in the tranflation from the Arabic, though the e-
nunciation of the propofition and the demonftra-
tion are there very much fpoiled. Jacobus Peleta-
rius, who was the firft, as far as I know, who took
notice of this error, gives alfo the right conftruction
in his edition of Euclid, after he had given the other which he
blames : He fays, he would not leave it out, becaufe it was fine,
and might fharpen one's genius to invent others like it ; whereas
there
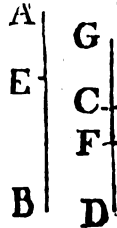
there is not the leaft difference between the two demonftrations, except a fingle word in the conftruction, which very probably has been owing to an unfkilful Librarian. Clavius likewife gives both the ways; but neither he nor Peletarius takes notice of the reafon why the one is preferable to the other.

P R O P.   VI.   B. V.

There are two cafes of this propofition, of which only the firft and fimpleft is demonftrated in the Greek: And it is probable Theon thought it was fufficient to give this one, fince he was to make ufe of neither of them in his mutilated edition of the 5th book; and he might as well have left out the other, as alfo the 5th propofition, for the fame reafon: The demonftration of the other cafe is now added, becaufe both of them, as alfo the 5th propofition, are neceffary to the demonftration of the 18th pro-pofition of this book.   The tranflation from the Arabic gives both cafes briefly.

P R O P.   A.   B. V.

This propofition is frequently ufed by geometers, and it is neceffary in the 25th prop. of this book, 31ft of the 6th, and 34th of the 11th, and 15th of the 12th book: It feems to have been taken out of the elements by Theon, becaufe it appeared evident enough to him, and others, who fubftitute the confufed and indiftinct idea the vulgar have of proportionals, in place of that accurate idea which is to be got from the 5th def. of this book.   Nor can there be any doubt that Eudoxus or Eu-clid gave it a place in the elements, when we fee the 7th and 9th of the fame book demonftrated, tho' they are quite as eafy and evident as this.   Alphonfus Borellus takes occafion from this propofition to cenfure the 5th definition of this book very feverely, but moft unjuftly: In p. 126. of his Euclid reftored, printed at Pifa in 1658, he fays,   " Nor can even this leaft de-" gree of knowledge be obtained from the forefaid property," viz. that which is contained in 5th def. 5.   " That, if four " magnitudes be proportionals, the third muft neceffarily be " greater than the fourth, when the firft is greater than the " fecond ; as Clavius acknowledges in the 16th prop. of the " 5th book of the elements."   But though Clavius makes no fuch acknowledgment exprefsly, he has given Borellus a han-dle to fay this of him; becaufe when Clavius, in the above ci-ted place, cenfures Commandine, and that very juftly, for de-monftrating this propofition by help of the 16th of the 5th; yet he himfelf gives no demonftration of it, but thinks it plain
from

from the nature of proportionals, as he writes in the end of the 14th and 16th prop. B. 5. of his edition, and is followed by Herigon in Schol. 1. prop. 14. B. 5. as if there was any nature of proportionals antecedent to that which is to be derived and underſtood from the definition of them : And indeed, though it is very eaſy to give a right demonſtration of it, no body, as far as I know, has given one, except the learned Dr Barrow, who, in anſwer to Borellus's objection, demonſtrates it indirectly, but very briefly and clearly, from the 5th definition, in the 322d page of his Lect. Mathem. from which definition it may alſo be eaſily demonſtrated directly : On which account we have placed it next to the propoſitions concerning equimultiples.

P R O P. B. B O O K V.

This alſo is eaſily deduced from the 5th def. B. 5. and therefore is placed next to the other ; for it was very ignorantly made a corollary from the 4th prop. of this book. See the note on that corollary.

P R O P. C. B. V.

This is frequently made uſe of by geometers, and is neceſſary to the 5th and 6th propoſitions of the 10th book. Clavius, in his notes ſubjoined to the 8th def. of book 5. demonſtrates it only in numbers, by help of ſome of the propoſitions of the 7th book, in order to demonſtrate the property contained in the 5th definition of the 5th book, when applied to numbers, from the property of proportionals contained in the 20th def. of the 7th book : And moſt of the commentators judge it difficult to prove that four magnitudes which are proportionals according to the 20th def. of 7th book, are alſo proportionals according to the 5th def. of 5th book. But this is eaſily made out, as follows.

First, If A, B, C, D be four magnitudes, ſuch that A is the ſame multiple, or the ſame part of B, which C is of D ; A, B, C, D are proportionals : This is demonſtrated in propoſition C.

Secondly, If AB contain the ſame parts of CD that EF does of GH ; in this caſe likewiſe AB is to CD, as EF to GH.

B F H
D
K L
A C E G M

Let
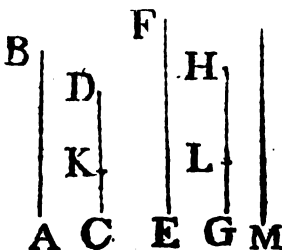
Let CK be a part of CD, and GL the same part of GH ;
and let AB be the same multiple of
CK, that EF is of GL : Therefore,
by prop. C. of 5th book, AB is to
CK, as EF to GL : And CD, GH
are equimultiples of CK, GL the
second and fourth ; wherefore, by
Cor prop. 4. Book 5. AB is to CD,
as EF to GH.
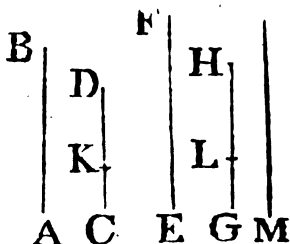
And if four magnitudes be pro-
portionals according to the 5th def.
of Book 5. they are also proportionals according to the 20th
def. of Book 7.

Firſt, If A be to B, as C to D ; then if A be any multiple
or part of B, C is the same multiple or part of D, by prop. D.
of B: 5.

Next, If AB be to CD, as EF to GH ; then if AB contains
any parts of CD, EF contains the same parts of GH : For let
CK be a part of CD, and GL the same part of GH, and let
AB be a multiple of CK ; EF is the same multiple of GL :
Take M the same multiple of GL that AB is of CK ; there-
fore by prop. C. of B. 5. AB is to CK, as M to GL ; and CD,
GH are equimultiples of CK, GL ; wherefore by Cor. prop.
4. B. 5. AB is to CD, as M to GH : And, by the hypothesis,
AB is to CD, as EF to GH ; therefore M is equal to EF by
prop. 9. Book 5. and consequently EF is the same multiple of
GL that AB is of CK.

P R O P. D. B. V.

This is not unfrequently uſed in the demonſtration of other
propoſitions, and is neceſſary in that of prop. 9. B. 6. It ſeems
Theon has left it out for the reaſon mentioned in the notes at
prop. A.

P R O P. VIII. B. V.

In the demonſtration of this, as it is now in the Greek,
there are two caſes, (ſee the demonſtration in Hervagius, or
Dr Gregory's edition), of which the firſt is that in which AE
is leſs than EB : and in this, it neceſſarily follows that HΘ
the multiple of EB is greater than ZH the same multiple of
AE, which laſt multiple, by the conſtruction, is greater than
Δ ; whence alſo HΘ muſt be greater than Δ : But, in the second
caſe, viz. that in which EB is leſs than AE, tho' ZH be greater
than Δ, yet HΘ may be leſs than the ſame Δ ; ſo that there
cannot be taken a multiple of Δ which is the firſt that is
greater

greater than K, or HΘ, becaufe Δ itfelf is greater than it : Up-
on this account, the author of this demonftration found it ne-
ceffary to change one part of the conftruction that was made
ufe of in the firft cafe : But he has, without any neceffity,
changed alfo another part of it, viz. when he orders to take

N that multiple of Δ which
is the firft that is greater than
ZH; for he might have taken
that multiple of Δ which is the
firft that is greater than HΘ,
or K, as was done in the firft
cafe : He likewife brings in
this K into the demonftration
of both cafes, without any rea-
fon ; for it ferves to no pur-
pofe but to lengthen the de-
monftration. There is alfo a
third cafe, which is not mentioned in this demonftration, viz.
that in which AE in the firft, or EB in the fecond of the two
other cafes, is greater than D ; and in this any equimultiples,
as the doubles, of AE, KB are to be taken, as is done in this
edition, where all the cafes are at once demonftrated : And from
this it is plain that Theon, or fome other unfkilful editor, has vi-
tiated this propofition.
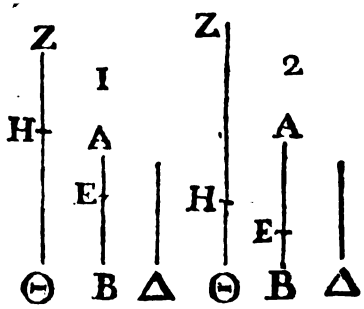
P R O P. IX. B. V.

Of this there is given a more explicit demonftration than that
which is now in the elements.

P R O P. X. B. V.

It was neceffary to give another demonftration of this pro-
pofition, becaufe that which is in the Greek and Latin, or o-
ther editions, is not legitimate : For the words *greater, the fame
or equal, leffer,* have a quite different meaning when applied
to magnitudes and ratios, as is plain from the 5th and 7th de-
finitions of Book 5. By the help of thefe let us examine the
demonftration of the 10th prop. which proceeds thus : " Let A
" have to C a greater ratio, than B to C : I fay that A is greater
" than B. For if it is not greater, it is either equal, or lefs.
" But A cannot be equal to B, becaufe then each of them
" would have the fame ratio to C; but they have not. There-
" fore A is not equal to B." The force of which reafoning is
this, if A had to C, the fame ratio that B has to C, then if
any equimultiples whatever of A and B be taken, and any
<div align="right">multiple</div>

multiple whatever of C; if the multiple of A be greater than the multiple of C, then, by the 5th def. of Book 5. the multiple of B is also greater than that of C; but, from the hypothesis that A has a greater ratio to C, than B has to C, there must, by the 7th def. of Book 5. be certain equimultiples of A and B, and some multiple of C such, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than the same multiple of C: And this proposition directly contradicts the preceding; wherefore A is not equal to B. The demonstration of the 10th prop. goes on thus: " But nei- " ther is A less than B; because then A would have a less ra- " tio to C, than B has to it: But it has not a less ratio, there- " fore A is not less than B," &c. Here it is said, that " A " would have a less ratio to C, than B has to C," or, which is the same thing, that B would have a greater ratio to C, than A to C; that is, by 7th def. Book 5. there must be some equimultiples of B and A, and some multiple of C such, that the multiple of B is greater than the multiple of C, but the multiple of A is not greater than it: And it ought to have been proved that this can never happen if the ratio of A to C be greater than the ratio of B to C; that is, it should have been proved, that, in this case, the multiple of A is always great- er than the multiple of C, whenever the multiple of B is greater than the multiple of C; for, when this is demonstrated, it will be evident that B cannot have a greater ratio to C, than A has to C, or, which is the same thing, that A cannot have a less ratio to C, than B has to C: But this is not at all proved in the 10th proposition; but if the 10th were once demonstrated, it would immediately follow from it, but cannot without it be easily demonstrated, as he that tries to do it will find. Where- fore the 10th proposition is not sufficiently demonstrated. And it seems that he who has given the demonstration of the 10th proposition as we now have it, instead of that which Eudoxus or Euclid had given, has been deceived in applying what is manifest, when understood of magnitudes, unto ratios, viz. that a magnitude cannot be both greater and less than another. That those things which are equal to the same are equal to one another, is a most evident axiom when understood of magnitudes; yet Euclid does not make use of it to infer that those ratios which are the same to the same ratio, are the same to one another; but explicitly demonstrates this in prop. 11. of Book 5. The demonstration we have given of the 10th prop. is

no

no doubt the fame with that of Eudoxus or Euclid, as it is immediately and directly derived from the definition of a greater ratio, viz. the 7. of the 5.

The above mentioned propofition, viz. If **A** have to **C** a greater ratio than B to C, and if of A and B there be taken certain equimultiples, and fome multiple of **C**; then if the multiple of B be greater than the multiple of C, the multiple of A is alfo greater than the fame, is thus demonftrated.

Let D, E be equimultiples of A, B, and F a multiple of C, fuch, that E the multiple of B is greater than F; D the multiple of A is alfo greater than F.

Becaufe A has a greater ratio to C, than B to C, A is greater than B, by the 10th prop. B. 5. therefore D the multiple of A is greater than E the fame multiple of B : And E is greater than F ; much more therefore D is greater than F.

A   C   B   C.
D   F   E   F
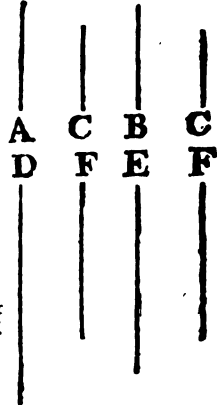
P R O P. XIII. B. V.

In Commandine's, Briggs's, and Gregory's tranflations, at the beginning of this demonftration, it is faid, " And the multi-
" ple of C is greater than the multiple of D ; but the multi-
" ple of E is not greater than the multiple of F ;" which words are a literal tranflation from the Greek : But the fenfe evidently requires that it be read, " fo that the multiple of C
" be greater than the multiple of D ; but the multiple of E be
" not greater than the multiple of F." And thus this place was reftored to the true reading in the firft editions of Commandine's Euclid, printed in 8vo at Oxford ; but in the later editions, at leaft in that of 1747, the error of the Greek text was kept in.

There is a corollary added to Prop. 13. as it is neceffary to the 20th and 21ft Prop. of this book, and is as ufeful as the propofition.

P R O P. XIV. B. V.

The two cafes of this, which are not in the Greek, are added ; the demonftration of them not being exactly the fame with that of the firft cafe.

X   P R O P.

P R O P. XVII. B. V.

The order of the words in a clause of this is changed to one more natural : As was also done in prop. 11.

P R O P. XVIII. B. V.

The demonstration of this is none of Euclid's, nor is it legitimate ; for it depends upon this hypothesis, that to any three magnitudes, two of which, at least, are of the same kind, there may be a fourth proportional ; which, if not proved, the demonstration now in the text is of no force : But this is assumed without any proof ; nor can it, as far as I am able to discern, be demonstrated by the propositions preceding this ; so far is it from deserving to be reckoned an axiom, as Clavius, after other commentators, would have it, at the end of the definitions of the 5th book. Euclid does not demonstrate it, nor does he shew how to find the fourth proportional, before the 12th prop. of the 6th book : And he never assumes any thing in the demonstration of a proposition, which he had not before demonstrated ; at least, he assumes nothing the existence of which is not evidently possible ; for a certain conclusion can never be deduced by the means of an uncertain proposition : Upon this account, we have given a legitimate demonstration of this proposition instead of that in the Greek and other editions, which very probably Theon, at least some other, has put in the place of Euclid's, because he thought it too prolix : And as the 17th prop. of which this 18th is the converse, is demonstrated by help of the 1st and 2d propositions of this book, so, in the demonstration now given of the 18th, the 5th prop. and both cases of the 6th are necessary, and these two propositions are the converses of the 1st and 2d. Now the 5th and 6th do not enter into the demonstration of any proposition in this book as we now have it : Nor can they be of use in any proposition of the Elements, except in this 18th, and this is a manifest proof, that Euclid made use of them in his demonstration of it, and that the demonstration now given, which is exactly the converse of that of the 17th, as it ought to be, differs nothing from that of Eudoxus or Euclid : For the 5th and 6th have undoubtedly been put into the 5th book for the sake of some propositions in it, as all the other propositions about equimultiples have been.

Hieronymus Saccherius, in his book named Euclides ab omni naevo vindicatus, printed at Milan ann. 1733, in 4to, acknowledges

knowledges this blemifh in the demonftration of the 18th, and
that he may remove it, and render the demonftration we now
have of it legitimate, he endeavours to demonftrate the follow-
ing propofition, which is in page 115. of his book, viz.

" Let A, B, C, D be four magnitudes, of which the two
" firft are of the one kind, and alfo the two others either of the
" fame kind with the two firft, or of fome other the fame
" kind with one another. I fay the ratio of the third C to the
" fourth D, is either equal to, or greater, or lefs than the ratio
" of the firft A to the fecond B."

And after two propofitions premifed as Lemmas, he proceeds
thus.

" Either among all the poffible equimultiples of the firft
" A, and of the third C, and, at the fame time, among all
" the poffible equimultiples of the fecond B, and of the
" fourth D, there can be found fome one multiple EF of the
" firft A, and one IK of the fecond B, that are equal to one
" another; and alfo (in the fame cafe) fome one multiple
" GH of the third C equal to LM the multiple of the fourth
" D, or fuch equality is no where to be found. If the firft
" cafe happen,
" [i. e. if fuch   A———   E————F
" equality is to
" be found] it is   B———   I————K
" manifeft from
" what is be-   C———   G————H
" fore demon-
" ftrated, that   D———   L—————M
" A is to B, as
" C to D; but if fuch fimultaneous equality be not to be
" found upon both fides, it will be found either upon one
" fide, as upon the fide of A [and B;] or it will be found
" upon neither fide; if the firft happen; therefore (from
" Euclid's definition of greater and leffer ratio foregoing)
" A has to B, a greater or lefs ratio than C to D; accor-
" ding as GH the multiple of the third C is lefs, or greater
" than LM the multiple of the fourth D: But if the fecond
" cafe happen; therefore upon the one fide, as upon the fide
" of A the firft and B the fecond, it may happen that the
" multiple EF, [viz. of the firft] may be lefs than IK the
" multiple of the fecond, while, on the contrary, upon the o-
" ther fide, [viz. of C and D] the multiple GH [of the third
" C] is greater than the other multiple LM [of the fourth
" D:] And then (from the fame definition of Euclid) the ra-
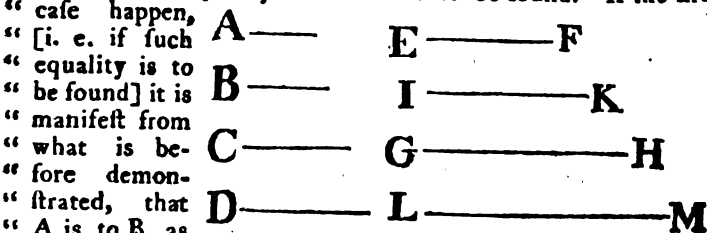
X 2

" tio

" tio of the firft A to the fecond B, is lefs than the ratio of the
" third C to the fourth D ; or on the contrary.

" Therefore the axiom, [i. e. the propofition before fet down],
" remains demonftrated," &c.

Not in the leaft ; but it remains ftill undemonftrated : For
what he fays may happen, may, in innumerable cafes, never
happen ; and therefore his demonftration does not hold : For
example, if A be the fide, and B the diameter of a fquare ;
and C the fide, and D the diameter of another fquare ; there
can in no cafe be any multiple of A equal to any of B ; nor
any one of C equal to one of D, as is well known ; and
yet it can never happen that when any multiple of A is greater
than a multiple of B, the multiple of C can be lefs than the mul-
tiple of D, nor when the multiple of A is lefs than that of B,
the multiple of C can be greater than that of D, viz. taking
equimultiples of A and C, and equimultiples of B and D : For
A, B, C, D are proportionals ; and fo if the multiple of A be
greater, &c. than that of B, fo muft that of C be greater, &c.
than that of D ; by 5th Def. b. 5.

The fame objection holds good againft the demonftration
which fome give of the 1ft prop. of the 6th book, which we have
made againft this of the 18th prop. becaufe it depends upon the
fame infufficient foundation with the other.

PROP. XIX. B. V.

A corollary is added to this, which is as frequently ufed as
the propofition itfelf. The corollary which is fubjoined to it
in the Greek, plainly fhews that the 5th book has been vitiated
by editors who were not geometers : For the converfion of
ratios does not depend upon this 19th, and the demonftration
which feveral of the commentators on Euclid give of conver-
fion, is not legitimate, as Clavius has rightly obferved, who
has given a good demonftration of it which we have put in pro-
pofition E ; but he makes it a corollary from the 19th, and be-
gins it with the words, " Hence it eafily follows," though it
does not at all follow from it.

PROP. XX. XXI. XXII. XXIII. XXIV. B. V.

The demonftrations of the 20th and 21ft propofitions are
fhorter than thofe Euclid gives of eafier propofitions, either
in the preceding, or following bocks : Wherefore it was pro-
per to make them more explicit, and the 22d and 23d propo-
fitions are, as they ought to be, extended to any number of
magnitudes :

magnitudes : And, in like manner, may the 24th be, as is taken notice of in a corollary ; and another corollary is added, as useful as the propofition, and the words, " any whatever" are fupplied near the end of prop. 23. which are wanting in the Greek text, and the tranflations from it.

In a paper writ by Philippus Naudaeus, and publifhed after his death, in the hiftory of the Royal Academy of Sciences of Berlin, anno 1745, page 50. the 23d prop. of the 5th book is cenfured as being obfcurely enunciated, and, becaufe of this, prolixly demonftrated : The enunciation there given is not Euclid's, but Tacquet's, as he acknowledges, which, though not fo well exprefled, is, upon the matter, the fame with that which is now in the Elements.   Nor is there any thing obfcure in it, though the author of the paper has fet down the proportionals in a difadvantageous order, by which it appears to be obfcure : But, no doubt, Euclid enunciated this 23d, as well as the 22d, fo as to extend it to any number of magnitudes, which, taken two and two, are proportionals, and not of fix only ; and to this general cafe the enunciation which Naudaeus gives, cannot be well applied.

The demonftration which is given of this 23d, in that paper, is quite wrong ; becaufe, if the proportional magnitudes be plane or folid figures, there can no rectangle (which he improperly calls a *product*) be conceived to be made by any two of them :   And if it fhould be faid, that in this cafe ftraight lines are to be taken which are proportional to the figures, the demonftration would this way become much longer than Euclid's : But, even though his demonftration had been right, who does not fee that it could not be made ufe of in the 5th book ?

P R O P. F, G, H, K. B. V.

Thefe propofitions are annexed to the 5th book, becaufe they are frequently made ufe of by both antient and modern geometers :   And in many cafes compound ratios cannot be brought into demonftration, without making ufe of them.

Whoever defires to fee the doctrine of ratios delivered in this 5th book folidly defended, and the arguments brought againft it by And. Tacquet, Alph. Borellus, and others, fully refuted, may read Dr Barrow's mathematical lectures, viz. the 7th and 8th of the year 1666.

The 5th book being thus corrected, I moft readily agree to what the learned Dr Barrow fays *, " That there is nothing
                              X 3   " in

* Page 336.

" in the whole body of the elements of a more subtile invention,
" nothing more solidly established, and more accurately handled,
" than the doctrine of proportionals."   And there is some
ground to hope, that geometers will think that this could not
have been said with as good reason, since Theon's time till the
present.

D E F. II. and V. of B. VI.

THE 2d definition does not seem to be Euclid's, but some
unskilful editor's: For there is no mention made by Eu-
clid, nor, as far as I know, by any other geometer, of recipro-
cal figures : It is obscurely expressed, which made it proper to
render it more distinct : It would be better to put the follow-
ing definition in place of it, viz.

D E F. II.

Two magnitudes are said to be reciprocally proportional to
two others, when one of the first is to one of the other magni-
tudes, as the remaining one of the last two is to the remaining
one of the first.
But the 5th definition, which, since Theon's time, has been
kept in the elements, to the great detriment of learners, is now
justly thrown out of them, for the reason given in the notes on
the 23d prop. of this book.

P R O P. I. and II. B. VI.

To the first of these a corollary is added, which is often used :
And the enunciation of the second is made more general.

P R O P. III. B. VI.

A second case of this, as useful as the first, is given in prop.
A; viz. the case in which the exterior angle of a triangle is bi-
sected by a straight line : The demonstration of it is very like to
that of the first case, and upon this account may, probably, have
been left out, as also the enunciation, by some unskilful editor :
At least, it is certain, that Pappus makes use of this case, as an
elementary proposition, without a demonstration of it, in prop.
39. of his 7th book of Mathematical Collections.

P R O P.

P R O P. VII. B. VI.

To this a cafe is added which occurs not unfrequently in de-monftrations.

P R O P. VIII. B. VI.

It feems plain that fome editor has changed the demonftra-tion that Euclid gave of this propofition : For, after he has de-monftrated, that the triangles are equiangular to one another, he particularly fhews that their fides about the equal angles are pro-portionals, as if this had not been done in the demonftration of the 4th prop. of this book : This fuperfluous part is not found in the tranflation from the Arabic, and is now left out.

P R O P. IX. B. VI.

This is demonftrated in a particular cafe, viz. that in which the third part of a ftraight line is required to be cut off ; which is not at all like Euclid's manner : Befides, the author of the demonftration, from four magnitudes being proportionals, con-cludes that the third of them is the fame multiple of the fourth, which the firft is of the fecond ; now, this is no where demon-ftrated in the 5th book, as we now have it : But the editor af-fumes it from the confufed notion which the vulgar have of pro-portionals : On this account, it was neceffary to give a general and legitimate demonftration of this proрofition.

P R O P. XVIII. B. VI.

The demonftration of this feems to be vitiated : For the pro-pofition is demonftrated only in the cafe of quadrilateral figures, without mentioning how it may be extended to figures of five or more fides : Befides, from two triangles being equi-angular, it is inferred, that a fide of the one is to the homolo-gous fide of the other, as another fide of the firft is to the fide homologous to it of the other, without permutation of the proportionals ; which is contrary to Euclid's manner, as is clear from the next propofition : And the fame fault occurs again in the conclufion, where the fides about the equal angles are not fhewn to be proportionals, by reafon of again neglect-ing permutation. On thefe accounts, a demonftration is gi-ven in Euclid's manner, like to that he makes ufe of in the

X 4   20th

20th Prop. of this book; and it is extended to five fided figures, by which it may be feen how to extend it to figures of any number of fides.

PROP. XXIII. B. VI.

Nothing is ufually reckoned more difficult in the elements of geometry by learners, than the doctrine of compound ratio, which Theon has rendered abfurd and ungeometrical, by fubftituting the 5th definition of the 6th book in place of the right definition, which without doubt Eudoxus or Euclid gave, in its proper place, after the definition of triplicate ratio, &c. in the 5th book. Theon's definition is this; a ratio is faid to be compounded of ratios ὅταν αἱ τῶν λόγων πηλικότητες ἐφ' ἑαυτὰς πολλαπλασιασθεῖσαι ποιῶσι τινα: Which Commandine thus tranflates " quando rationum quantitates inter fe multi-" plicatae aliquam efficiunt rationem;" that is, when the quantities of the ratios being multiplied by one another make a certain ratio. Dr Wallis tranflates the word πηλικότητες " ra-" tionum exponentes," the exponents of the ratios : And Dr Gregory renders the laft words of the definition by " illius fa-" cit quantitatem," makes the quantity of that ratio: But in whatever fenfe the " quantities," or " exponents of the ra-" tios," and their " multiplication" be taken, the definition will be ungeometrical and ufelefs : For there can be no multi-plication but by a number: Now the quantity or exponent of a ratio (according as Eutocius in his Comment. on prop. 4. book 2. of Arch. de Sph. et Cyl. and the moderns explain that term) is the number which multiplied into the confequent term of a ratio produces the antecedent, or, which is the fame thing, the number which arifes by dividing the antecedent by the con-fequent ; but there are many ratios fuch, that no number can arife from the divifion of the antecedent by the confequent ; ex. gr. the ratio which the diameter of a fquare has to the fide of it; and the ratio which the circumference of a circle has to its diameter, and fuch like. Befides, that there is not the leaft mention made of this definition in the writings of Eu-clid, Archimedes, Apollonius, or other antients, tho' they fre-quently make ufe of compound ratio : And in this 23d prop. of the 6th book, where compound ratio is firft mentioned, there is not one word which can relate to this definition, though here, if in any place, it was neceffary to be brought in ; but the right definition is exprefsly cited in thefe words : " But the " ratio of K to M is compounded of the ratio of K to L, " and

" and of the ratio of L to M." This definition therefore of Theon is quite ufelefs and abfurd: For that Theon brought it into the elements can fcarce be doubted; as it is to be found in his commentary upon Ptolomy's Μιγαλη Συνταξις, page 62. where he alfo gives a childifh explication of it, as agreeing only to fuch ratios as can be expreffed by numbers; and from this place the definition and explication have been exactly copied and prefixed to the definitions of the 6th book, as appears from Hervagius's edition: But Zambertus and Commandine, in their Latin tranflations, fubjoin the fame to thefe definitions. Neither Campanus, nor, as it feems, the Arabic manufcripts, from which he made his tranflation, have this definition. Clavius, in his obfervations upon it, rightly judges that the definition of compound ratio might have been made after the fame manner in which the definitions of duplicate and triplicate ratio are given, viz. " That as in feveral magni-
" tudes that are continual proportionals, Euclid named the
" ratio of the firft to the third, the duplicate ratio of the
" firft to the fecond; and the ratio of the firft to the fourth,
" the triplicate ratio of the firft to the fecond that is, the
" ratio compounded of two or three intermediate ratios that
" are equal to one another, and fo on; fo, in like manner, if
" there be feveral magnitudes of the fame kind, following one
" another, which are not continual proportionals, the firft is
" faid to have to the laft the ratio compounded of all the in-
" termediate ratios,——only for this reafon, that thefe inter-
" mediate ratios are interpofed betwixt the two extremes, viz.
" the firft and laft magnitudes; even as, in the 10th definition
" of the 5th book, the ratio of the firft to the third was called
" the duplicate ratio, merely upon account of two ratios be-
" ing interpofed betwixt the extremes, that are equal to one
" another: So that there is no difference betwixt this com-
" pounding of ratios, and the duplication or triplication of
" them which are defined in the 5th book, but that in the du-
" plication, triplication, &c. of ratios, all the interpofed ratios
" are equal to one another; whereas, in the compounding of
" ratios, it is not neceffary that the intermediate ratios fhould
" be equal to one another." Alfo Mr Edmund Scarburgh, in his Englifh tranflation of the firft fix books, page 238. 266. exprefsly affirms, that the 5th definition of the 6th book, is fuppofititious, and that the true definition of compound ratio
is

is contained in the 10th definition of the 5th book, viz. the
definition of duplicate ratio, or to be underſtood from it, to
wit, in the ſame manner as Clavius has explained it in the pre-
ceding citation. Yet theſe, and the reſt of the moderns, do
notwithſtanding retain this 5th def. of the 6th book, and illu-
ſtrate and explain it by long commentaries, when they ought ra-
ther to have taken it quite away from the elements.

For, by comparing def. 5. book 6. with prop. 5. book 8.
it will clearly appear that this definition has been put into the
elements in place of the right one which has been taken out
of them : Becauſe, in prop. 5. book 8. it is demonſtrated that
the plane number of which the ſides are C, D has to the plane
number of which the ſides are E, Z, (ſee Hergavius's or
Gregory's edition), the ratio which is compounded of the ra-
tios of their ſides ; that is, of the ratios of C to E, and D to
Z : And by def. 5. book 6. and the explication given of it by
all the commentators, the ratio which is compounded of the ra-
tios of C to E, and D to Z, is the ratio of the product made
by the multiplication of the antecedents C, D to the product
of the conſequents E, Z ; that is, the ratio of the plane number
of which the ſides are C, D to the plane number of which
the ſides are E, Z. Wherefore the propoſition which is the 5th
def. of book 6. is the very ſame with the 5th prop. of book 8.
and therefore it ought neceſſarily to be cancelled in one of theſe
places; becauſe it is abſurd that the ſame propoſition ſhould
ſtand as a definition in one place of the elements, and be de-
monſtrated in another place of them. Now, there is do doubt
that prop. 5. book 8. ſhould have a place in the elements, as
the ſame thing is demonſtrated in it concerning plane num-
bers, which is demonſtrated in prop. 23d book 6. of equiangu-
lar parallelograms ; wherefore def. 5. book 6. ought not to be
in the elements. And from this it is evident that this definition
is not Euclid's, but Theon's, or ſome other unſkilful geometer's.

But no body, as far as I know, has hitherto ſhown the true
uſe of compound ratio, or for what purpoſe it has been in-
troduced into geometry ; for every propoſition in which
compound ratio is made uſe of, may without it be both enun-
ciated and demonſtrated. Now the uſe of compound ratio
conſiſts wholly in this, that by means of it, circumlocutions
may be avoided, and thereby propoſitions may be more brief-
ly either enunciated or demonſtrated, or both may be done ;
for inſtance, if this 23d propoſition of the ſixth book were to
be enunciated, without mentioning compound ratio, it might
be

be done as follows. If two parallelograms be equiangular, and if as a fide of the firft to a fide of the fecond, fo any affumed ftraight line be made to a fecond ftraight line; and as the o-ther fide of the firft to the other fide of the fecond, fo the fe-cond ftraight line be made to a third. The firft parallelogram is to the fecond, as the firft ftraight line to the third. And the demonftration would be exactly the fame as we now have it. But the antient geometers, when they obferved this enuncia-tion could be made fhorter, by giving a name to the ratio which the firft ftraight line has to the laft, by which name the intermediate ratios might likewife be fignified, of the firft to the fecond, and of the fecond to the third, and fo on, if there were more of them, they called this ratio of the firft to the laft, the ratio compounded of the ratios of the firft to the fe-cond, and of the fecond to the third ftraight line; that is, in the prefent example, of the ratios which are the fame with the ratios of the fides, and by this they expreffed the propo-fition more briefly thus: If there be two equiangular paralle-lograms, they have to one another the ratio which is the fame with that which is compounded of ratios that are the fame with the ratios of the fides. Which is fhorter than the preceding enunciation, but has precifely the fame meaning. Or yet fhorter thus: Equiangular parallelograms have to one another the ratio which is the fame with that which is com-pounded of the ratios of their fides. And thefe two enuncia-tions, the firft efpecially, agree to the demonftration which is now in the Greek. The propofition may be more briefly de-monftrated, as Candalla does, thus: Let ABCD, CEFG be two equiangular parallelograms, and complete the parallelo-gram CDHG; then, becaufe there are three parallelograms AC, CH, CF, the firft AC (by the definition of compound ratio) has to the third CF, the ratio which is compounded of the ratio of the firft AC to the fecond CH, and of the ratio of CH to the third CF; but the parallelogram AC is to the pa-rallelogram CH, as the ftraight line BC to CG; and the parallelogram CH is to CF, as the ftraight line CD is to CE; therefore the parallelogram AC has to CF the ratio which is compounded of ratios that are the fame with the ratios of the fides. And to this demonftration agrees the enun-ciation which is at prefent in the text, viz. Equiangular paralle-lograms have to one another the ratio which is compounded of

the
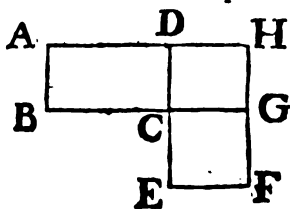
the ratios of the fides : For the vulgar reading, " which is com= " pounded of their fides," is abfurd.   But, in this edition, we have kept the demonftration which is in the Greek text, though not fo fhort as Candalla's ; becaufe the way of finding the ratio which is compounded of the ratios of the fides, that is, of find- ing the ratio of the parallelograms, is fhewn in that, but not in Candalla's demonftration ; whereby beginners may learn, in like cafes, how to find the ratio which is compounded of two or more given ratios.

From what has been faid, it may be obferved, that in any magnitudes whatever of the fame kind A, B, C, D, &c. the ratio compounded of the ratios of the firft to the .fecond, of the fecond to the third, and fo on to the laft, is only a name or expreffion by which the ratio which the firft A has to the laft D is fignified, and by which at the fame time the ratios of all the magnitudes A to B, B to C, C to D from the firft to the laft, to one another, whether they be the fame, or be not the fame, are indicated ; as in magnitudes which are continual proportionals A, B, C, D, &c. the duplicate ratio of the firft to the fecond is only a name, or expreffion by which the ratio of the firft A to the third C is fignified, and by which, at the fame time, is fhown that there are two ratios of the magni- tudes from the firft to the laft, viz. of the firft A to the fe- cond B, and of the fecond B to the third or laft C, which are the fame with one another ; and the triplicate ratio of the firft to the fecond is a name or expreffion by which the ratio of the firft A to the fourth D is fignified, and by which, at the fame time, is fhown that there are three ratios of the magni- tudes from the firft to the laft, viz. of the firft A to the fe- cond B, and of B to the third C, and of C to the fourth or laft D, which are all the fame with one another ; and fo in the cafe of any other multiplicate ratios.   And that this is the right explication of the meaning of thefe ratios is plain from the definitions of duplicate and triplicate ratio in which Euclid makes ufe of the word λίγεται, is faid to be, or is called ; which word, he, no doubt, made ufe of alfo in the definition of compound ratio, which Theon, or fome other, has expun= ged from the elements ; for the very fame word is ftill retained in the wrong definition of compound ratio, which is now the 5th of the 6th book : But in the citation of thefe definitions it is fometimes retained, as in the demonftration of prop. ·19.

book

book 6. " the firſt is ſaid to have, ἔχειν λέγεται, to the third the
" duplicate ratio," &c. which is wrong tranſlated by Comman-
dine and others, " has" inſtead of " is ſaid to have ;" and
ſometimes it is left out, as in the demonſtration of prop. 33.
of the 11th Book, in which we find " the firſt has, ἔχει, to the
" third the triplicate ratio ;" but without doubt ἔχει, " has,"
in this place ſignifies the ſame as ἔχειν λέγεται, is ſaid to have :
So likewiſe in prop. 23. B. 6. we find this citation, " but the
" ratio of K to M is compounded, σύγκειται, of the ratio of
" K to L, and the ratio of L to M," which is a ſhorter way of
expreſſing the ſame thing, which, according to the definition,
ought to have been expreſſed by συγκεῖσθαι λέκεται, is ſaid to be
compounded.

From theſe remarks, together with the propoſitions ſubjoined
to the 5th book, all that is found concerning compound ratio,
either in the antient or modern geometers, may be underſtood
and explained.

P R O P. XXIV. B. VI.

It ſeems that ſome unſkilful editor has made up this demon-
ſtration as we now have it, out of two others ; one of which may
be made from the 2d prop. and the other from the 4th of this
book : For after he has, from the 2d of this book, and compo-
ſition and permutation, demonſtrated that the ſides about the
angle common to the two parallelograms are proportionals, he
might have immediately concluded that the ſides about the other
equal angles were proportionals, viz. from prop. 34. B. 1. and
prop. 7. Book 5. This he does not, but proceeds to ſhow that
the triangles and parallelograms are equiangular ; and in a te-
dious way, by help of prop. 4. of this book, and the 22d of
Book 5. deduces the ſame concluſion : From which it is plain
that this ill compoſed demonſtration is not Euclid's : Theſe ſu-
perfluous things are now left out, and a more ſimple demonſtra-
tion is given from the 4th prop. of this book, the ſame which
is in the tranſlation from the Arabic, by help of the 2d prop.
and compoſition ; but in this the author neglects permutation,
and does not ſhow the parallelograms to be equiangular, as is
proper to do for the ſake of beginners.

P R O P. XXV. B. VI.

It is very evident that the demonſtration which Euclid had
given of this propoſition has been vitiated by ſome unſkilful
hand : For, after this editor had demonſtrated that " as the
" rectilineal figure ABC is to the rectilineal KGH, ſo is the
" parallelogram

" parallelogram BE to the parallelogram EF;" nothing more
should have been added but this, " and the rectilineal figure
" ABC is equal to the parallelogram BE; therefore the recti-
" lineal KGH is equal to the parallelogram EF," viz. from
prop. 14. book 5. But betwixt these two sentences he has in-
serted this; " wherefore, by permutation, as the rectilineal fi-
" gure ABC to the parallelogram BE, so is the rectilineal KGH
" to the parallelogram EF;" by which, it is plain, he thought
it was not so evident to conclude that the second of four pro-
portionals is equal to the fourth from the equality of the first
and third, which is a thing demonstrated in the 14th prop. of
B. 5. as to conclude that the third is equal to the fourth, from
the equality of the first and second, which is no where demon-
strated in the elements as we now have them: But though this
proposition, viz. the third of four proportionals is equal to the
fourth, if the first be equal to the second, had been given in
the elements by Euclid, as very probably it was, yet he would
not have made use of it in this place; because, as was said, the
conclusion could have been immediately deduced without this
superfluous step by permutation: This we have shown at the
greater length, both because it affords a certain proof of the
vitiation of the text of Euclid; for the very same blunder is
found twice in the Greek text of prop. 23. book 11. and twice
in prop. 2. b. 12. and in the 5. 11. 12. and 18th of that book;
in which places of book 12. except the last of them, it is rightly
left out in the Oxford edition of Commandine's translation:
And also that geometers may beware of making use of permu-
tation in the like cases; for the moderns not unfrequently com-
mit this mistake, and among others Commandine himself in his
commentary on prop. 5. book 3. p. 6. b. of Pappus Alexandri-
nus, and in other places: The vulgar notion of proportionals
has, it seems, preoccupied many so much, that they do not suf-
ficiently understand the true nature of them.

Besides, though the rectilineal figure ABC, to which another
is to be made similar, may be of any kind whatever; yet in the
demonstration the Greek text has " triangle" instead of rectili-
" neal figure," which error is corrected in the above-named
Oxford edition.

P R O P. XXVII. B. VI.

The second case of this has 'αλλάς, otherwise, prefixed to
it, as if it was a different demonstration, which probably has
been done by some unskilful Librarian. Dr Gregory has right-
ly

ly left it out : The fcheme of this fecond cafe ought to be
marked with the fame letters of the alphabet which are in the
fcheme of the firft, as is now done.

P R O P. XXVIII. and XXIX. B. VI.

Thefe two problems, to the firft of which the 27th prop. is
neceffary, are the moft general and ufeful of all in the elements,
and are moft frequently made ufe of by the antient geometers
in the folution of other problems ; and therefore are very igno-
rantly left out by Tacquet and Dechales in their editions of the
elements, who pretend that they are fcarce of any ufe : The ca-
fes of thefe problems, wherein it is required to apply a rectan-
gle which fhall be equal to a given fquare, to a given ftraight
line, either deficient or exceeding by a fquare ; as alfo to apply
a rectangle which fhall be equal to another given, to a given
ftraight line, deficient or exceeding by a fquare, are very often
made ufe of by geometers : And, on this account, it is thought
proper, for the fake of beginners, to give their conftructions,
as follows.

1. To apply a rectangle which fhall be equal to a given fquare,
to a given ftraight line, deficient by a fquare : But the given
fquare muft not be greater than that upon the half of the given
line.

. Let AB be the given ftraight line, and let the fquare upon
the given ftraight line C be that to which the rectangle to be
applied muft be equal, and this fquare, by the determination, is
not greater than that upon half of the ftraight line AB.

Bifect AB in D, and if the fquare upon AD be equal to
the fquare upon C, the thing required is done : But if it be not
equal to it, AD muft be
greater than C, acccording
to the determination : Draw
DE at right angles to AB,
and make it equal to C ;
produce ED to F, fo that
EF be equal to AD or DB,
and from the centre E, at
the diftance EF, defcribe a
circle meeting AB in G,
and upon GB defcribe the fquare GBKH, and complete the
rectangle AGHL ; alfo join EG : And becaufe AB is bifected
in D, the rectangle AG, GB together with the fquare of DG
is equal * to (the fquare of DB, that is, of EF or EG, that is, a 5.
to)
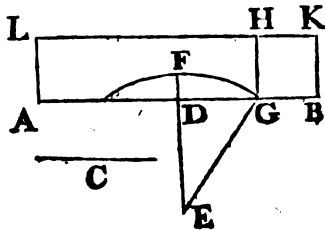
to) the fquares of ED, DG: Take away the fquare of DG from each of thefe equals; therefore the remaining rectangle AG, GB is equal to the fquare of ED, that is, of C: But the rectangle AG, GB is the rectangle AH, becaufe GH is equal to GB; therefore the rectangle AH is equal to the given fquare upon the ftraight line C. Wherefore the rectangle AH, equal to the given fquare upon C, has been applied to the given ftraight line AB, deficient by the fquare GK. Which was to be done.

2. To apply a rectangle which fhall be equal to a given fquare, to a given ftraight line, exceeding by a fquare.

Let AB be the given ftraight line, and let the fquare upon the given ftraigbt line C be that to which the rectangle to be applied muft be equal.

Bifect AB in D, and draw BE at right angles to it, fo that BE be equal to C; and having joined DE, from the centre D at the diftance DE defcribe a circle meeting AB produced in G; upon BG defcribe the fquare BGHK, and complete the rect-angle AGHL. And becaufe AB is bifected in D, and produced to G, the rectangle AG, GB together with the fquare of DB

a 6. 2. is equal ª to (the fquare of DG, or DE, that is, to) the fquares of EB, BD. From each of thefe equals take the fquare of DB; therefore the remaining rectangle AG, GB is equal to the fquare of BE, that is, to the fquare upon C. But the rectangle AG, GB is the rectangle AH, becaufe GH is equal to GB: Therefore the rectangle AH is equal to the fquare upon C. Wherefore the rectangle AH, equal to the given fquare upon C, has been applied to the given ftraight line AB, exceeding by the fquare GK. Which was to be done.
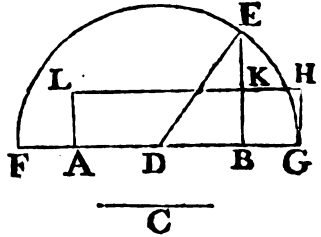
3. To apply a rectangle to a given ftraight line which fhall be equal to a given rectangle, and be deficient by a fquare. But the given rectangle muft not be greater than the fquare upon the half of the given ftraight line.

Let AB be the given ftraight line, and let the given rectangle be that which is contained by the ftraight lines C, D which is not greater than the fquare upon the half of AB; it is required to apply to AB a rectangle equal to the rectangle C, D, defi-cient by a fquare.

Draw

Draw AE, BF at right angles to AB, upon the same side of it, and make AE equal to C, and BF to D: Join EF and bisect it in G; and from the centre G, at the distance GE, describe a circle meeting AE again in H; join HF and draw GK parallel to it, and GL parallel to AE meeting AB in L.

Because the angle EHF in a femicircle is equal to the right angle EAB, AB and HF are parallels, and AH and BF are parallels; wherefore AH is equal to BF, and the rectangle EA, AH equal to the rectangle EA, BF, that is to the rectangle C, D: And because EG, GF are equal to one another, and AE, LG, BF parallels; therefore AL and LB are equal; also EK is equal to KH [a], and the rectangle C, D from the determination, is not greater than the fquare of AL the half of AB; wherefore the rectangle EA, AH is not greater than the fquare of AL, that is of KG: Add to each the fquare of KE; therefore the fquare [b] of AK is not greater than the fquares of EK, KG, that is, than the fquare of EG; and confequently the ftraight line AK or GL is not greater than GE. Now, if GE be equal to GL, the circle EHF touches AB in L, and therefore the fquare of AL is [c] equal to the rectangle EA, AH, that is, to the given rectangle C, D; and that which was required is done: But if EG, GL be unequal, EG muft be the greater; and therefore the circle EHF cuts the ftraight line AB; let it cut it

in the points M, N, and upon NB defcribe the fquare NBOP, and complete the rectangle ANPQ: Becaufe ML is equal to [d] LN, and it has been proved that AL is equal to LB; therefore AM is equal to NB, and the rectangle AN, NB equal to the rectangle NA, AM, that is, to the rectangle [e] EA, AH or the rectangle C, D: But the rectangle AN, NB is the rectangle AP, becaufe PN is equal to NB: Therefore the rectangle AP is equal to the rectangle C, D; and the rectangle AP equal to the given rectangle C, D has been applied to the given ftraight line AB, deficient by the fquare BP. Which was to be done.
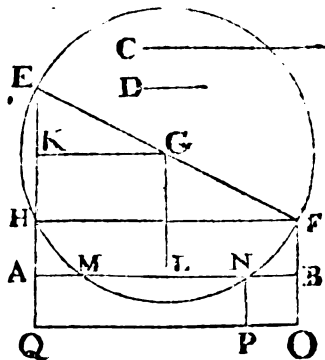
Y   4. To

4. To apply a rectangle to a given ftraight line that fhall be equal to a given rectangle, exceeding by a fquare.

Let AB be the given ftraight line, and the rectangle C, D the given rectangle, it is required to apply a rectangle to AB equal to C, D, exceeding by a fquare.

Draw AE, BF at right angles to AB, on the contrary fides of it, and make AE equal to C, and BF equal to D: Join EF, and bifect it in G; and from the centre G, at the diftance GE, defcribe a circle meeting AE again in H; join HF, and draw GL parallel to AE; let the circle meet AB produced in M, N, and upon BN defcribe the fquare NBOP, and complete the rectangle ANPQ; becaufe the angle EHF in a femicircle is equal to the right angle EAB. AB and HF are parallels, and therefore AH and BF are e- qual, and the rectangle EA, AH equal to the rectangle
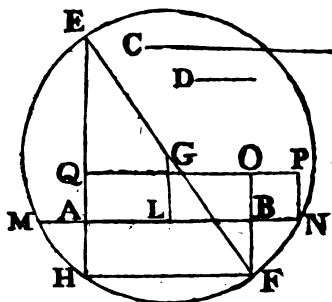
EA, BF, that is, to the rectangle C, D: And becaufe ML is equal to LN, and AL to LB, therefore MA is equal to BN, and a 35. 3. the rectangle AN, NB to MA, AN, that is, [a] to the rectangle EA, AH, or the rectangle C, D: Therefore the rectangle AN, NB, that is, AP, is equal to the rectangle C, D; and to the gi- ven ftraight line AB the rectangle AP has been applied equal to the given rectangle C, D, exceeding by the fquare BP. Which was to be done.

Willebrordus Snellius was the firft, as far as I know, who gave thefe conftructions of the 3d and 4th problems in his Ap- pollonius Batavus: And afterwards the learned Dr Halley gave them in the Scholium of the 18th prop. of the 8th book of A- pollonius's conics reftored by him.

The 3d problem is otherwife enunciated thus: To cut a given ftraight line AB in the point N, fo as to make the rect- angle AN, NB equal to a given fpace: Or, which is the fame thing, having given AB the fum of the fides of a rect- angle, and the magnitude of it being likewife given, to find its fides.

And the 4th problem is the fame with this, To find a point N in

N in the given ſtraight line AB produced, ſo as to make the
rectangle AN, NB equal to a given ſpace : Or, which is the
ſame thing, having given AB the difference of the ſides of a
rectangle, and the magnitude of it, to find the ſides.

P R O P. XXXI. B. VI.

In the demonſtration of this, the inverſion of proportionals is
twice neglected, and is now added, that the concluſion may be
legitimately made by help of the 24th prop. of b. 5. as Clavius
had done.

P R O P. XXXII. B. VI.

The enunciation of the preceding 26th prop. is not general e-
nough ; becauſe not only two ſimilar parallelograms that have
an angle common to both, are about the ſame diameter ; but
likewiſe two ſimilar parallelograms that have vertically oppoſite
angles, have their diameters in the ſame ſtraight line : But there
ſeems to have been another, and that a direct demonſtration of
theſe caſes, to which this 32d propoſition was needful : And the
32d may be otherwiſe and ſomething more briefly demonſtrated
as follows.

P R O P. XXXII. B. VI.

If two triangles which have two ſides of the one, &c.

Let GAF, HFC be two triangles which have two ſides AG,
GF, proportional to the two ſides FH, HC, viz. AG to GF, as
FH to HC ; and let AG be paral-
lel to FH, and GF to HC ; AF
and FC are in a ſtraight line.

Draw CK parallel ᵃ to FH, and
let it meet GF produced in K :
Becauſe AG, KC are each of them
parallel to FH, they are parallel ᵇ
to one another, and therefore the
alternate angles AGF, FKC are e-
qual : And AG is to GF, as (FH to HC, that is ᶜ) CK to KF ;
wherefore the triangles AGF, CKF are equiangular ᵈ, and the
angle AFG equal to the angle CFK : But GFK is a ſtraight line,
therefore AF and FC are in a ſtraight line ᵉ.

a 31. 1.

b 30. 1.

c 34. 1.
d 6. 6.

e 14. 1.

The 26th prop. is demonſtrated from the 32d, as follows.

If two ſimilar and ſimilarly placed parallelograms have an an-
gle common to both, or vertically oppoſite angles ; their diame-
ters are in the ſame ſtraight line.

Y 2
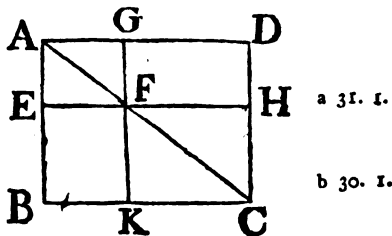
Firſt,

Book VI.   Firſt, Let the parallelograms ABCD, AEFG have the angle
BAD common to both, and be ſimilar, and ſimilarly placed;
ABCD, AEFG are about the ſame diameter.

Produce EF, GF, to H, K, and join FA, FC: Then be-
cauſe the parallelograms ABCD, AEFG are ſimilar, DA
is to AB, as GA to AE; where-

a Cor. 19.   fore the remainder DG is ᵃ to the
5.   remainder EB, as GA to AE: But
DG is equal to FH, EB to HC,
and AE to GF: Therefore as FH
to HC, ſo is AG to GF; and
FH, HC are parallel to AG, GF;
and the triangles AGF, FHC are
joined at one angle, in the point

b 32. 6.   F; wherefore AF, FC are in the ſame ſtraight line ᵇ.
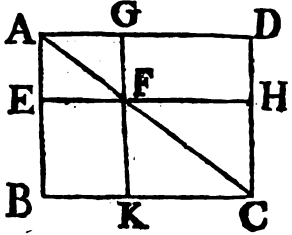
Next, Let the parallelograms KFHC, GFEA, which are ſimi-
lar and ſimilarly placed, have their angles KFH, GFE vertically
oppoſite; their diameters AF, FC are in the ſame ſtraight line.

Becauſe AG, GF are parallel to FH, HC; and that AG
is to GF, as FH to HC; therefore AF, FC are in the ſame
ſtraight line ᵇ.

P R O P. XXXIII. B. VI.

The words " becauſe they are at the centre," are left out, as
the addition of ſome unſkilful hand.

In the Greek, as alſo in the Latin tranſlation, the words
ἀ ʼτυχι, " any whatever," are left out in the demonſtration of
both parts of the propoſition, and are now added as quite neceſ-
ſary; and, in the demonſtration of the ſecond part, where the
triangle BGC is proved to be equal to CGK, the illative par-
ticle ἄρα in the Greek text ought to be omitted.

The ſecond part of the propoſition is an addition of Theon's,
as he tells us in his commentary on Ptolomy's Μεγάλη Συντάξις,
p. 50.

P R O P. B. C. D. B. VI.

Theſe three propoſitions are added, becauſe they are frequent-
ly made uſe of by geometers.

D E F.

D E F. IX. and XI. B. XI.

THE fimilitude of plane figures is defined from the e-
quality of their angles, and the proportionality of the
fides about the equal angles; for from the proportionality of
the fides only, or only from the equality of the angles, the
fimilitude of the figures does not follow, except in the cafe
when the figures are triangles : The fimilar pofition of the fides,
which contain the figures, to one another, depending partly
upon each of thefe : And, for the fame reafon, thofe are fi-
milar folid figures which have all their folid angles equal, each
to each, and are contained by the fame number of fimilar
plane figures : For there are fome folid figures contained by fi-
milar plane figures, of the fame number, and even of the
fame magnitude, that are neither fimilar nor equal, as fhall
be demonftrated after the notes on the 10th definition : Upon
this account it was neceffary to amend the definition of fimi-
lar folid figures, and to place the definition of a folid angle
before it : And from this and the 10th definition, it is fuffi-
ciently plain how much the elements have been fpoiled by un-
fkilful editors.

D E F. X. B. XI.

Since the meaning of the word " equal" is known and
eftablifhed before it comes to be ufed in this definition;
therefore the propofition which is the 10th definition of this
book, is a theorem, the truth or falfehood of which ought to
be demonftrated, not affumed; fo that Theon, or fome o-
ther Editor, has ignorantly turned a theorem which ought
to be demonftrated into this 10th definition : That figures are
fimilar, ought to be proved from the definition of fimilar
figures; that they are equal ought to be demonftrated from
the axiom, " Magnitudes that wholly coincide, are equal
" to one another;" or from prop. A. of book 5. or the 9th
prop. or the 14th of the fame book, from one of which the
equality of all kind of figures muft ultimately be deduced.
In the preceding books, Euclid has given no definition of e-
qual figures, and it is certain he did not give this : For what is
called the 1ft def. of the 3d book, is really a theorem in
which thefe circles are faid to be equal, that have the ftraight
lines from their centres to the circumferences equal, which is
plain, from the definition of a circle; and therefore has by

<div align="center">Y 3</div>   fome

Book XI. fome editor been improperly placed among the definitions. The
equality of figures ought not to be defined, but demonſtrated :
Therefore, though it were true, that ſolid figures contained by
the ſame number of ſimilar and equal plane figures are equal to
one another, yet he would juſtly deſerve to be blamed who
ſhould make a definition of this propoſition which ought to be
demonſtrated. But if this propoſition be not true, muſt it not
be confeſſed, that geometers have, for theſe thirteen hundred
years, been miſtaken in this elementary matter ? And this ſhould
teach us modeſty, and to acknowledge how little, through the
weakneſs of our minds, we are able to prevent miſtakes even in
the principles of ſciences which are juſtly reckoned amongſt the
moſt certain ; for that the propoſition is not univerſally true,
can be ſhewn by many examples : The following is ſufficient.

Let there be any plane rectilineal figure, as the triangle
a 12. 11. ABC, and from a point D within it draw ᵃ the ſtraight line
DE at right angles to the plane ABC; in DE take DE, DF
equal to one another, upon the oppoſite ſides of the plane,
and let G be any point in EF; join DA, DB, DC; EA,
EB, EC; FA, FB, FC; GA, GB, GC: Becauſe the ſtraight
line EDF is at right angles to the plane ABC, it makes right
angles with DA, DB, DC which it meets in that plane ; and
in the triangles EDB, FDB, ED and DB are equal to FD and
DB, each to each, and they contain right angles; therefore

b 4. 1. the baſe EB is equal ᵇ
to the baſe FB ; in the
ſame manner EA is e-
qual to FA, and EC to
FC: And in the triangles
EBA, FBA, EB, BA
are equal to FB, BA,
and the baſe EA is e-
qual to the baſe FA ;
wherefore the angle
e 8. 2. EBA is equal ᶜ to the
angle FBA, and the tri-
angle EBA equal ᵇ to
the triangle FBA, and
the other angles equal to
the other angles ; there-
d { 4. 6. fore theſe triangles are
{ 1. def.
{ 6. ſimilar ᵈ: In the ſame manner the triangle EBC is ſimilar to
               the
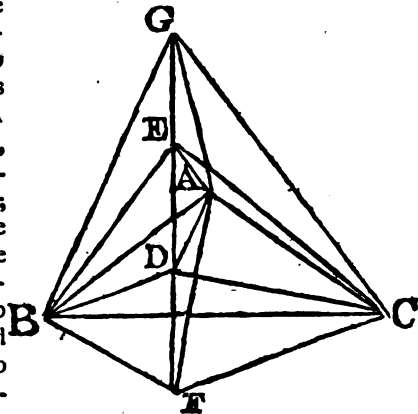
the triangle FBC, and the triangle EAC to FAC; therefore there are two solid figures each of which is contained by six triangles, one of them by three triangles, the common vertex of which is the point G, and their bases the straight lines AB, BC, CA; and by three other triangles the common vertex of which is the point E, and their basis the same lines AB, BC, CA: The other solid is contained by the same three triangles the common vertex of which is G, and their bases AB, BC, CA; and by three other triangles of which the common vertex is the point F, and their bases the same straight lines AB, BC, CA: Now the three triangles GAB, GBC, GCA are common to both solids, and the three others EAB, EBC, ECA of the first solid have been shown equal and similar to the three others FAB, FBC, FCA of the other solid, each to each; therefore these two solids are contained by the same number of equal and similar planes: But that they are not equal is manifest, because the first of them is contained in the other: Therefore it is not universally true that solids are equal which are contained by the same number of equal and similar planes.

Cor. From this it appears that two unequal solid angles may be contained by the same number of equal plane angles.

For the solid angle at B, which is contained by the four plane angles EBA, EBC, GBA, GBC is not equal to the solid angle at the same point B which is contained by the four plane angles FBA, FBC, GBA, GBC; for this last contains the other: And each of them is contained by four plane angles, which are equal to one another, each to each, or are the self same; as has been proved: And indeed there may be innumerable solid angles all unequal to one another, which are each of them contained by plane angles that are equal to one another, each to each: It is likewise manifest that the before-mentioned solids are not similar, since their solid angles are not all equal.

And that there may be innumerable solid angles all unequal to one another, which are each of them contained by the same plane angles disposed in the same order, will be plain from the three following propositions.

PROP. I. PROBLEM.

Three magnitudes, A, B, C being given, to find a fourth such, that every three shall be greater than the remaining one. Let D be the fourth; therefore D must be less than A, B, C

C together: Of the three A, B C, let A be that which is not lefs than either of the two B and C: And firſt, let B and C together be not lefs than A: therefore B, C, D together are greater than A; and becauſe A is not lefs than B; A, C, D together are greater than B: In the like manner A, B, D together are greater than C: Wherefore in the caſe in which B and C together are not lefs than A, any magnitude D which is lefs than A, B, C together will anſwer the problem.

But if B and C together be lefs than A; then, becauſe it is required that B, C, D together be greater than A, from each of theſe taking away B, C, the remaining one D muſt be greater than the exceſs of A above B and C: Take therefore any magnitude D which is lefs than A, B, C together, but greater than the exceſs of A above B and C: Then B, C, D together are greater than A; and becauſe A is greater than either B or C, much more will A and D, together with either of the two B, C be greater than the other: And, by the conſtruction, A, B, C are together greater than D.

Cor. If beſides it be required, that A and B together ſhall not be lefs than C and D together; the exceſs of A and B together above C muſt not be lefs than D, that is, D muſt not be greater than that exceſs.

P R O P. II. P R O B L E M.

Four magnitudes A, B, C, D being given, of which A and B together are not lefs than C and D together, and ſuch that any three of them whatever are greater than the fourth; it is required to find a fifth magnitude E ſuch, that any two of the three A, B, E ſhall be greater than the third, and alſo that any two of the three C, D, E ſhall be greater than the third. Let A be not lefs than B: And C not lefs than D.

Firſt, Let the exceſs of C above D be not lefs than the exceſs of A above B: It is plain that a magnitude E can be taken which is lefs than the ſum of C and D, but greater than the exceſs of C above D; let it be taken; then E is greater likewiſe than the exceſs of A above B; wherefore E and B together are greater than A; and A is not lefs than B; therefore A and E together are greater than B: And, by the hypotheſis, A and B together are not lefs than C and D together, and C and D together are greater than E; therefore likewiſe A and B are greater than E.

<div align="right">But</div>

But let the excefs of A above B be greater than the excefs of Book XI.
C above D : And becaufe, by the hypothefis, the three B, C,
D are together greater than the fourth A ; C and D together
are greater than the exeefs of A above B : Therefore a magni-
tude may be taken which is lefs than C and D together, but
greater than the excefs of A above B. Let this magnitude be
E ; and becaufe E is greater than the excefs of A above B, B
together with E is greater than A : And, as in the preceding
cafe, it may be fhown that A together with E is greater than B,
and that A together with B is greater than E : Therefore, in
each of the cafes, it has been fhown that any two of the three
A, B, E are greater than the third.

And becaufe in each of the cafes E is greater than the excefs
of C above D, E together with D is greater than C ; and, by
the hypothefis, C is not lefs than D ; therefore E together with
C is greater than D ; and, by the conftruction, C and D toge-
ther are greater than E : Therefore any two of the three, C, D,
E are greater than the third.

P R O P. III.   T H E O R E M.

There may be innumerable folid angles all unequal to one an-
other, each of which is contained by the fame four plane an-
gles, placed in the fame order.

Take three plane angles, A, B, C, of which A is not lefs
than either of the other two, and fuch, that A and B toge-
ther are lefs than two right angles ; and by problem 1. and
its corollary, find a fourth angle D fuch, that any three what-
ever of the angles A, B, C, D be greater than the remaining
angle, and fuch, that A and B together be not lefs than C
and D together : And by problem 2. find a fifth angle E fuch,
that any two of the angles A, B, E be greater than the third,
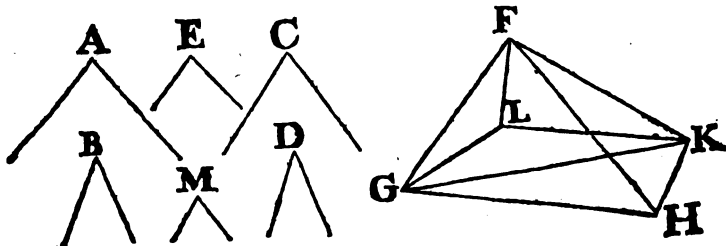
and alfo that any two of the angles C, D, E be greater than
the

the third: And becaufe A and B together are lefs than two
right angles, the double of A and B together is lefs than four
right angles: But A and B together are greater than the angle
E; wherefore the double of A, B together is greater than
the three angles A, B, E together, which three are confe-
quently lefs than four right angles; and every two of the
fame angles A, B, E are greater than the third; therefore,
by prop. 23. 11. a folid angle may be made contained by three
plane angles equal to the angles A, B, E, each to each. Let
this be the angle F contained by the three plane angles GFH,
HFK, GFK which are equal to the angles A, B, E, each to
each: And becaufe the angles C, D together are not greater
than the angles A, B together, therefore the angles C, D, E
are not greater than the angles A, B, E: But thefe laft three are
lefs than four right angles, as has been demonftrated; where-
fore alfo the angles C, D, E are together lefs than four right
angles, and every two of them are greater than the third; there-
fore a folid angle may be made which fhall be contained by three
a 23. 11. plane angles equal to the angles C, D, E, each to each[a]: And
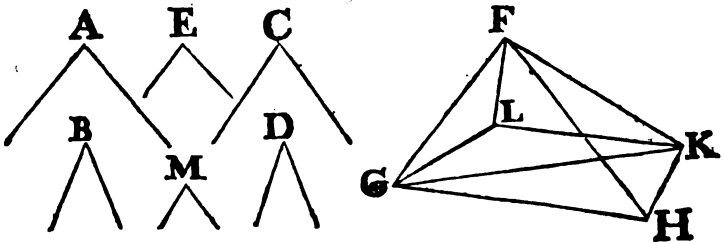
by prop. 26. 11. at the point F in the ftraight line FG a folid
angle may be made equal to that which is contained by the
three plane angles that are equal to the angles C, D, E: Let
this be made, and let the angle GFK, which is equal to E, be
one of the three; and let KFL, GFL be the other two which
are equal to the angles, C, D, each to each. Thus there is a
folid angle conftituted at the point F contained by the four plane
angles GFH, HFK, KFL, GFL which are equal to the angles
A, B, C, D, each to each.

Again, Find another angle M fuch, that every two of the
three angles A, B, M be greater than the third, and alfo
every two of the three C, D, M be greater than the third:
And,

And, as in the preceding part, it may be demonftrated that the three A, B, M are lefs than four right angles, as alfo that the three C, D, M are lefs than four right angles. Make therefore [a] a folid angle at N contained by the three plane angles ONP, PNQ, ONQ, which are equal to A, B, M, each to each : And by prop. 26. 11. make at the
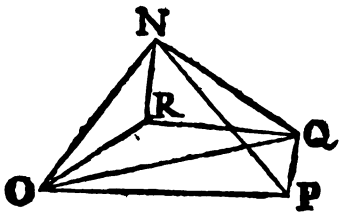
point N in the ftraight line ON a folid angle contained by three plane angles of which one is the angle ONQ equal to M, and the other two are the angles QNR, ONR which are equal to the angles C, D, each to each. Thus, at the point N, there is a folid angle contained by the four plane angles ONP, PNQ, QNR, ONR which are equal to the angles A, B, C, D, each to each. And that the two folid angles at the points F, N, each of which is contained by the above named four plane angles, are not equal to one another, or that they cannot coincide, will be plain by confidering that the angles GFK, ONQ; that is, the angles E, M, are unequal by the conftruction ; and therefore the ftraight lines GF, FK cannot coincide with ON, NQ, nor confequently can the folid angles, which therefore are unequal.

And becaufe from the four plane angles A, B, C, D, there can be found innumerable other angles that will ferve the fame purpofe with the angles E and M ; it is plain that innumerable other folid angles may be conftituted which are each contained by the fame four plane angles, and all of them unequal to one another, Q. E. D.

And from this it appears that Clavius and other authors are miftaken, who affert that thofe folid angles are equal which are contained by the fame number of plane angles that are equal to one another, each to each. Alfo it is plain that the 26th prop. of book 11. is by no means fufficiently demonftrated, becaufe the equality of two folid angles, whereof each is contained by three plane angles which are equal to one another, each to each, is only affumed, and not demonftrated.

P R O P.

P R O P. I. B. XI.

The words at the end of this, " for a ftraight line cannot
" meet a ftraight line in more than one point," are left out, as an
addition by fome unfkilful hand ; for this is to be demonftrated,
not affumed.

Mr. Thomas Simpfon, in his notes at the end of the 2d edition
of his elements of geometry, p. 262. after repeating the words
of this note, adds, " Now, can it poffibly fhow any want of
" fkill in an editor (he means Euclid or Theon) to refer to an
" axiom which Euclid himfelf hath laid down, book 1. No 14."
he means Barrow's Euclid, for it is the 10th in the Greek, " and
" not to have demonftrated, what no man can demonftrate?"
But all that in this cafe can follow from that axiom is, that, if
two ftraight lines could meet each other in two points, the parts
of them betwixt thefe points muft coincide, and fo they would
have a fegment betwixt thefe points common to both. Now, as
it has not been fhown in Euclid, that they cannot have a com-
mon fegment, this does not prove that they cannot meet in two
points, from which their not having a common fegment is de-
duced in the Greek edition : But, on the contrary, becaufe they
cannot have a common fegment, as is fhown in Cor. of 11th
prop. book 1. of 4to edition, it follows plainly that they cannot
meet in two points, which the remarker fays no man can de-
monftrate.

Mr Simpfon, in the fame notes, p. 265. juftly obferves, that
in the corollary of prop. 11. book 1. 4to edit. the ftraight lines
AB, BD, BC, are fuppofed to be all in the fame plane, which
cannot be affumed in 1ft prop. book 11. This, foon after the
4to edition was publifhed, I obferved and correfted as it is now
in this edition : He is miftaken in thinking the 10th axiom he
mentions here to be Euclid's ; it is none of Euclid's, but is the
10th in Dr Barrow's edition, who had it from Herigon's Curfus,
vol. 1. and in place of it the corollary of 10th prop. book 1. was
added.

P R O P. II. B. XI.

This propofition feems to have been changed and vitiated by
fome editor ; for all the figures defined in the 1ft book of the
elements, and among them triangles, are, by the hypothefis,
plane figures ; that is, fuch as are defcribed in a plane ; where-
fore the fecond part of the enunciation needs no demonftration.
Befides, a convex fuperficies may be terminated by three ftraight
lines

lines meeting one another: The thing that fhould have been demonftrated is, that two, or three ftraight lines, that meet one another, are in one plane. And as this is not fufficiently done, the enunciation and demonftration are changed into thofe now put into the text.

P R O P. III. B. XI.

In this propofition the following words near to the end of it are left out, viz. " therefore DEB, DFB are not ftraight lines ; " in the like manner it may be demonftrated that there can be " no other ftraight line between the points D, B :" Becaufe from this, that two lines include a fpace, it only follows that one of them is not a ftraight line : And the force of the argument lies in this, viz. if the common fection of the planes be not a ftraight line, then two ftraight lines could include a fpace, which is abfurd ; therefore the common fection is a ftraight line.

P R O P. IV. B. XI.

The words " and the triangle AED to the triangle BEC" are omitted, becaufe the whole conclufion of the 4th prop. b. 1. has been fo often repeated in the preceding books, it was needlefs to repeat it here.

P R O P. V. B. XI.

In this, near to the end, ἐπιπέδω, ought to be left out in the Greek text: And the word " plane" is rightly left out in the Oxford edition of Commandine's tranflation.

P R O P. VII. B. XI.

This propofition has been put into this book by fome unfkilful editor, as is evident from this, that ftraight lines which are drawn from one point to another in a plane, are, in the preceding books, fuppofed to be in that plane : And if they were not, fome demonftrations in which one ftraight line is fuppofed to meet another would not be conclufive, becaufe thefe lines would not meet one another : For inftance, in prop. 30 b. 1. the ftraight line GK would not meet EF, if GK were not in the plane in which are the parallels AB, CD, and in which, by hypothefis, the ftraight line EF is : Befides, this 7th propofition is demonftrated by the preceding 3. in which the very thing which is propofed to be demonftrated in the 7th, is twice affumed, viz. that the ftraight line drawn from one point to another in a plane, is in that plane ; and the fame thing is affumed in the preceding 6th prop. in which the ftraight line which

which joins the points B, D that are in the plane to which AB and CD are at right angles, is suppoſed to be in that plane: And the 7th, of which another demonſtration is given, is kept in the book merely to preſerve the number of the propoſitions ; for it is evident from the 7th and 35th definitions of the 1ſt book, though it had not been in the elements.

P R O P.   VIII.   B. XI.

In the Greek, and in Commandine's and Dr Gregory's tranſ- lations, near to the end of this propoſition, are the following words: " But DC is in the plane through BA, AD," inſtead of which, in the Oxford edition of Commandine's tranſlation, is rightly put " but DC is in the plane through BD, DA :" But all the editions have the following words, viz. " becauſe AB, " BD are in the plane thro' BD, DA, and DC is in the plane " in which are AB, BD," which are manifeſtly corrupted, or have been added to the text ; for there was not the leaſt neceſſi- ty to go ſo far about to ſhew that DC is in the ſame plane in which are BD, DA, becauſe it immediately follows from prop. 7. preceding, that BD, DA are in the plane in which are the parallels AB, CD : Therefore, inſtead of theſe words, there ought only to be " becauſe all three are in the plane in which " are the parallels AB, CD."

P R O P.   XV.   B. XI.

After the words, " and becauſe BA is parallel to GH," the following are added, " for each of them is parallel to DE, and " are not both in the ſame plane with it," as being manifeſtly forgotten to be put into the text.

P R O P.   XVI.   B. XI.

In this, near to the end, inſtead of the words, " but ſtraight " lines which meet neither way" ought to be read, " but ſtraight " lines in the ſame plane which produced meet neither way :" Becauſe, though in citing this definition in prop. 27. book 1. it was not neceſſary to mention the words, " in the ſame plane," all the ſtraight lines in the books preceding this being in the ſame plane ; yet here it was quite neceſſary.

P R O P.   XX.   B. XI.

In this, near the beginning, are the words, " But if not, " let BAC be the greater :" But the angle BAC may happen to be equal to one of the other two : Wherefore this place ſhould be

be read thus, " But if not, let the angle BAC be not lefs than
" either of the other two, but greater than DAB."

At the end of this propofition it is faid, " in the fame man-
" ner it may be demonftrated," though there is no need of any
demonftration ; becaufe the angle BAC being not lefs than ei-
ther of the other two, it is evident that BAC together with one
of them is greater than the other.

P R O P. XXII. B. XI.

And likewife in this, near the beginning, it is faid, " But
" if not, let the angles at B, E, H be unequal, and let the
" angle at B be greater than either of thofe at E, H :" Which
words manifeftly fhow this place to be vitiated, becaufe the an-
gle at B may be equal to one of the other two. They ought
therefore to be read thus, " But if not, let the angles at B, E,
" H be unequal, and let the angle at B be not lefs than either
" of the other two at E, H : Therefore the ftraight line AC is
" not lefs than either of the two DF, GK."

P R O P. XXIII. B. XI.

The demonftration of this is made fomething fhorter, by not
repeating in the third cafe the things which were demonftrated
in the firft ; and by making ufe of the conftruction which Cam-
panus has given ; but he does not demonftrate the fecond and
third cafes : The conftruction and demonftration of the third
cafe are made a little more fimple than in the Greek text.

P R O P. XXIV. B. XI.

The word " fimilar" is added to the enunciation of this pro-
pofition, becaufe the planes containing the folids which are to
be demonftrated to be equal to one another, in the 25th propo-
fition, ought to be fimilar and equal, that the equality of the
folids may be inferred from prop. C. of this book : And, in the
Oxford edition of Commandine's tranflation, a corollary is ad-
ded to prop. 24. to fhow that the parallelograms mentioned in
this propofition are fimilar, that the equality of the folids in
prop. 25. may be deduced from the 10th def. of book 11.

P R O P. XXV. and XXVI. B. XI.

In the 25th prop. folid figures which are contained by the
fame number of fimilar and equal plane figures, are fuppofed
to be equal to one another. And it feems that Theon, or fome
other

other editor, that he might fave himſelf the trouble of demon-
ſtrating the ſolid figures mentioned in this propoſition to be
equal to one another, has inſerted the 10th def. of this book,
to ſerve inſtead of a demonſtration ; which was very ignorantly
done.

Likewife in the 26th prop. two ſolid angles are ſuppoſed to
be equal : If each of them be contained by three plane angles
which are equal to one another, each to each. And it is ſtrange
enough, that none of the commentators on Euclid have, as far
as I know, perceived that ſomething is wanting in the demon-
ſtrations of theſe two propoſitions. Clavius, indeed, in a note
upon the 11th def. of this book, affirms, that it is evident that
thoſe ſolid angles are equal which are contained by the ſame
number of plane angles, equal to one another, each to each,
becauſe they will coincide, if they be conceived to be placed
within one another ; but this is ſaid without any proof, nor is it
always true, except when the ſolid angles are contained by three
plane angles only, which are equal to one another, each to
each : And in this caſe the propoſition is the ſame with this,
that two ſpherical triangles that are equilateral to one another,
are alſo equiangular to one another, and can coincide ; which
ought not to be granted without a demonſtration. Euclid does
not aſſume this in the caſe of rectilineal triangles, but demon-
ſtrates in prop. 8. book 1. that triangles which are equilateral
to one another are alſo equiangular to one another ; and from
this their total equality appears by prop. 4. book 1. And Me-
nelaus, in the 4th prop. of his 1ſt book of ſpherics, explicitly
demonſtrates that ſpherical triangles which are mutually equi-
lateral, are alſo equiangular to one another ; from which it is
eaſy to ſhow that they muſt coincide, providing they have their
ſides diſpoſed in the ſame order and ſituation.

To ſupply theſe defects, it was neceſſary to add the three pro-
poſitions marked A, B, C to this book. For the 25th, 26th, and
28th propoſitions of it, and conſequently eight others, viz. the
27th, 31ſt, 32d, 33d, 34th, 36th, 37th, and 40th of the ſame,
which depend upon them, have hitherto ſtood upon an infirm
foundation ; as alſo, the 8th, 12th, Cor. of 17th and 18th of
12th book, which depend upon the 9th definition. For it has
been ſhown in the notes on def. 10. of this book, that ſolid
figures which are contained by the ſame number of ſimilar and
equal plane figures, as alſo ſolid angles that are contained by
the ſame number of equal plane angles, are not always equal to
one another,

I:

It is to be obferved that Tacquet, in his Euclid, defines equal folid angles to be fuch, " as being put within one another do " coincide:" But this is an axiom, not a definition; for it is true of all magnitudes whatever. He made this ufelefs definition, that by it he might demonftrate the 36th prop. of this book, without the help of the 35th of the fame: Concerning which demonftration, fee the note upon prop. 36.

P R O P. XXVIII. B. XI.

In this it ought to have been demonftrated, not affumed, that the diagonals are in one plane. Clavius has fupplied this defect.

P R O P. XXIX. B. XI.

There are three cafes of this propofition; the firft is, when the two parallelograms oppofite to the bafe AB have a fide common to both; the fecond is, when thefe parallelograms are feparated from one another; and the third, when there is a part of them common to both; and to this laft only, the demonftration that has hitherto been in the elements does agree. The firft cafe is immediately deduced from the preceding 28th prop. which feems for this purpofe to have been premifed to this 29th, for it is neceffary to none but to it, and to the 40th of this book, as we now have it, to which laft it would, without doubt, have been premifed, if Euclid had not made ufe of it in the 29th; but fome unfkilful editor has taken it away from the elements, and has mutilated Euclid's demonftration of the other two cafes, which is now reftored, and ferves for both at once.

P R O P. XXX. B. XI.

In the demonftration of this, the oppofite planes of the folid CP, in the figure in this edition, that is, of the folid CO in Commandine's figure, are not proved to be parallel; which it is proper to do for the fake of learners.

P R O P. XXXI. B. XI.

There are two cafes of this propofition; the firft is, when the infifting ftraight lines are at right angles to the bafes; the other, when they are not: The firft cafe is divided again into two others, one of which is, when the bafes are equiangular parallelograms; the other, when they are not equiangular:

Z   The

Book XI. The Greek editor makes no mention of the firſt of theſe twe
laſt caſes, but has inſerted the demonſtration of it as a part of
that of the other: And therefore ſhould have taken notice of
it in a corollary; but we thought it better to give theſe two caſes
ſeparately: The demonſtration alſo is made ſomething ſhorter
by following the way Euclid has made uſe of in prop. 14. book
6. Beſides, in the demonſtration of the caſe in which the in-
ſiſting ſtraight lines are not at right angles to the baſes, the edi-
tor does not prove that the ſolids deſcribed in the conſtruction
are parallelepipeds, which it is not to be thought that Euclid ne-
glected: Alſo the words, " of which the inſiſting ſtraight lines
" are not in the ſame ſtraight lines," have been added by ſome
unſkilful hand; for they may be in the ſame ſtraight lines.

PROP. XXXII. B. XI.

The editor has forgot to order the parallelogram FH to be
applied in the angle FGH equal to the angle LCG, which is
neceſſary. Clavius has ſupplied this.

Alſo, in the conſtruction, it is required to complete the
ſolid of which the baſe is FH, and altitude the ſame with that
of the ſolid CD: But this does not determine the ſolid to be
completed, ſince there may be innumerable ſolids upon the
ſame baſe, and of the ſame altitude: It ought therefore to be
ſaid " complete the ſolid of which the baſe is FH, and one of
" its inſiſting ſtraight lines is FD:" The ſame correction muſt
be made in the following propoſition 33.

PROP. D. B. XI.

It is very probable that Euclid gave this propoſition a place in
the elements, ſince he gave the like propoſition concerning equi-
angular parallelograms in the 23d b. 6.

PROP. XXXIV. B. XI.

In this the words, ὧν αἱ ἐφιστῶσαι ἐκ εἰσιν ἐπι τῶν αυτῶν εὐθειῶν,
" of which the inſiſting ſtraight lines are not in the ſame
" ſtraight lines," are thrice repeated; but theſe words ought
either to be left out, as they are by Clavius, or, in place of them,
ought to be put, " whether the inſiſting ſtraight lines be, or be
" not, in the ſame ſtraight lines:" For the other caſe is with-
out any reaſon excluded; alſo the words, ὧν τα ὕψη, of which
the

" the altitudes," are twice put for ὧν αἱ ἐφιστᾶσαι, " of which
" the infifting ftraight lines ;" which is a plain miftake : For the
altitude is always at right angles to the bafe.

P R O P. XXXV. B. XI.

The angles ABH, DEM are demonftrated to be right angles
in a fhorter way than in the Greek ; and in the fame way ACH,
DFM may be demonftrated to be right angles: Alfo the repe-
tition of the fame demonftration, which begins with " in the
" fame manner," is left out, as it was probably added to the
text by fome editor ; for the words, " in like manner we may
" demonftrate," are not inferted except when the demonftration
is not given, or when it is fomething different from the other,
if it be given, as in prop. 26. of this book. Companus has not
this repetition.

We have given another demonftration of the corollary, be-
fides the one in the original, by help of which the following
36th prop. may be demonftrated without the 35th.

P R O P. XXXVI. B. XI.

Tacquet in his Euclid demonftrates this propofition without
the help of the 35th ; but it is plain, that the folids mentioned
in the Greek text in the enunciation of the propofition as equi-
angular, are fuch that their folid angles are contained by three
plane angles equal to one another, each to each ; as is evident
from the conftruction. Now Tacquet does not demonftrate,
but affumes thefe folid angles to be equal to another ; for
he fuppofes the folids to be already made, and does not give the
conftruction by which they are made : But, by the fecond de-
monftration of the preceding corollary, his demonftration is
rendered legitimate likewife in the cafe where the folids are con-
ftructed as in the text.

P R O P. XXXVII. B. XI.

In this it is affumed that the ratios which are triplicate of thofe
ratios which are the fame with one another, are likewife the
fame with one another ; and that thofe ratios are the fame with
one another, of which the triplicate ratios are the fame with one
another ; but this ought not to be granted without a demonftra-
tion ; nor did Euclid affume the firft and eafieft of thefe two
propofitions, but demonftrated it in the cafe of duplicate ratios,
in the 22d prop. book 6. On this account, another demonftra-
tion is given of this propofition like to that which Euclid gives
in prop. 22. book 6. as Clavius has done.

<center>Z 2</center>

<center>P R O P.</center>

PROP. XXXVIII. B. XI.

When it is required to draw a perpendicular from a point in one plane which is at right angles to another plane, unto this laſt plane, it is done by drawing a perpendicular from the point to the common ſection of the planes; for this perpendicular will be perpendicular to the plane, by Def. 4. of this book: And it would be fooliſh in this caſe to do it by the 11th prop. of the ſame: But Euclid [a], Apollonius, and other geometers, when they have occaſion for this problem, direct a perpendicular to be drawn from the point to the plane, and conclude that it will fall upon the common ſection of the planes, becauſe this is the very ſame thing as if they had made uſe of the conſtruction above mentioned, and then concluded that the ſtraight line muſt be perpendicular to the plane; but is expreſſed in fewer words: Some editor, not perceiving this, thought it was neceſſary to add this propoſition, which can never be of any uſe to the 11th book, and its being near to the end among propoſitions with which it has no connection, is a mark of its having been added to the text.

PROP. XXXIX. B. XI.

In this it is ſuppoſed, that the ſtraight lines which biſect the ſides of the oppoſite planes, are in one plane, which ought to have been demonſtrated; as is now done.

B. XII.

THE learned Mr Moor, profeſſor of Greek in the Univerſity of Glaſgow, obſerved to me, that it plainly appears from Archimedes's epiſtle to Dofitheus, prefixed to his books of the Sphere and Cylinder, which epiſtle he has reſtored from antient manuſcripts, that Eudoxus was the author of the chief propoſitions in this 12th book.

PROP. II. B. XII.

At the beginning of this it is ſaid, " if it be not ſo, the ſquare " of BD ſhall be to the ſquare of FH, as the circle ABCD is " to ſome ſpace either leſs than the circle EFGH, or greater " than it:" And the like is to be found near to the end of this propoſition, as alſo in prop. 5. 11. 12. 18. of this book: Concerning

cerning which, it is to be obferved, that, in the demonftration of theorems, it is fufficient, in this and the like cafes, that a thing made ufe of in the reafoning can poffibly exift, providing this be evident, though it cannot be exhibited or found by a geometrical conftruction: So, in this place, it is affumed, that there may be a fourth proportional to thefe three magnitudes, viz. the fquares of BD, FH, and the circle ABCD; becaufe it is evident that there is fome fquare equal to the circle ABCD, though it cannot be found geometrically; and to the three rectilineal figures, viz. the fquares of BD, FH, and the fqure which is equal to the circle ABCD, there is a fourth fquare proportional; becaufe to the three ftraight lines which are their fides, there is a fourth ftraight line proportional*, and this fourth fquare, or a fpace equal to it, is the fpace which in this propofition is denoted by the letter S: And the like is to be underftood in the other places above cited: And it is probable that this has been fhewn by Euclid, but left out by fome editor; for the lemma which fome unfkiltul hand has added to this propofition explains nothing of it.

* 12. 6.

P R O P. III. B. XII.

In the Greek text and the tranflations, it is faid, " and " becaufe the two ftraight lines BA, AC which meet one an- " other," &c. here the angles BAC, KHL are demonftrated to be equal to one another by 10th prop. b. 11. which had been done before: Becaufe the triangle EAG was proved to be fimilar to the triangle KHL: This repetition is left out, and the triangles BAC, KHL are proved to be fimilar in a fhorter way by prop. 21. b. 6.

P R O P. IV. B. XII.

A few things in this are more fully explained than in the Greek text.

P R O P. V. B. XII.

In this, near to the end, are the words, ὡς ἔμπροσθεν ἐδειχθη, " as was before fhown," and the fame are found again in the end of prop. 18. of this book; but the demonftration referred to, except it be the ufelefs lemma annexed to the 2d prop. is no where in thefe elements, and has been perhaps left out by fome editor who has forgot to cancel thofe words alfo.

PROP. VI. B. XII.

A shorter demonstration is given of this; and that which
is in the Greek text may be made shorter by a step than it is:
For the author of it makes use of the 22d prop. of b. 5. twice:
Whereas once would have served his purpose; because that pro-
position extends to any number of magnitudes which are pro-
portionals taken two and two, as well as to three which are pro-
portional to other three.

COR. PROP. VIII. B. XII.

The demonstration of this is imperfect, because it is not
shown, that the triangular pyramids into which those upon mul-
tangular bases are divided, are similar to one another, as ought
necessarily to have been done, and is done in the like case in
prop. 12. of this book: The full demonstration of the corollary
is as follows.

Upon the polygonal bases ABCDE, FGHKL, let there be si-
milar and similarly situated pyramids which have the points M,
N for their vertices: The pyramid ABCDEM has to the pyra-
mid FGHKLN the triplicate ratio of that which the side AB
has to the homologus side FG.

a 20. 6.   Let the polygons be divided into the triangles ABE, EBC,
b 11. def.   ECD; FGL, LGH, LHK, which are similar [a] each to each:
11.   And because the pyramids are similar, therefore [b] the triangle
c 4. 6.   EAM is similar to the triangle LFN, and the triangle ABM
             to FGN: Wherefore [c] ME is to EA, as NL to LF; and as AE
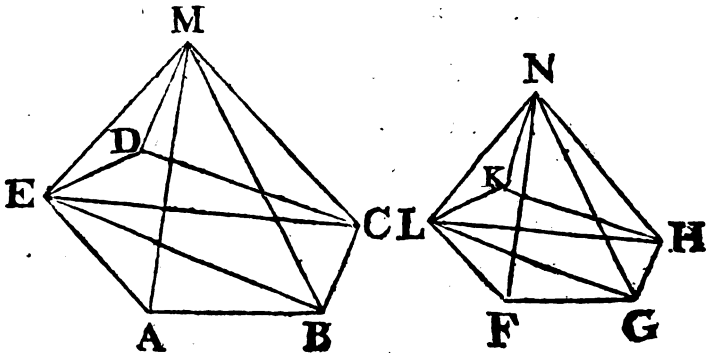
to EB, so is FL to LG, because the triangles EAB, LFG are
similar; therefore, ex aequali, as ME to EB, so is NL to LG:
                                                              In

In like manner it may be fhewn that EB is to BM, as LG to GN; therefore, again, *ex aequali*, as EM to MB, fo is LN to NG: Wherefore the triangles EMB, LNG having their fides proportionals are ^d equiangular, and fimilar to one another: Therefore the pyramids which have the triangles EAB, LFG for their bafes, and the points M, N for their vertices, are fimilar ^b to one another, for their folid angles are ^e equal, and the folids themfelves are contained by the fame number of fimilar planes: In the fame manner the pyramid EBCM may be fhewn to be fimilar to the pyramid LGHN, and the pyramid ECDM to LHKN: And becaufe the pyramids EABM, LFGN are fimilar, and have triangular bafes, the pyramid EABM has ^f to LFGN the triplicate ratio of that which EB has to the homologous fide LG. And, in the fame manner, the pyramid EBCM has to the pyramid LGHN the triplicate ratio of that which EB has to LG: Therefore, as the pyramid EABM is to the pyramid LFGN, fo is the pyramid EBCM to the pyramid LGHN: In like manner, as the pyramid EBCM is to LGHN, fo is the pyramid ECDM to LHKN: And as one of the antecedents is to one of the confequents, fo are all the antecedents to all the confequents: Therefore as the pyramid EABM to the pyramid LFGN, fo is the whole pyramid ABCDEM to the whole pyramid FGHKLN: And the pyramid EABM has to the pyramid LFGN the triplicate ratio of that which AB has to FG; therefore the whole pyramid has to the whole pyramid the triplicate ratio of that which AB has to the homologous fide FG. Q. E. D.

d s. 6.

b 11. def.
11.
c b. 11.

f 8. 12.

P R O P. XI. and XII. B. XII.

The order of the letters of the alphabet is not obferved in thefe two propofitions, according to Euclid's manner, and is now reftored: By which means, the firft part of prop. 12. may be demonftrated in the fame words with the firft part of prop. 11.; on this account the demonftration of that firft part is left out, and affumed from prop. 11.

P R O P. XIII. B. XII.

In this propofition the common fection of a plane parallel to the bafes of a cylinder, with the cylinder itfelf, is fuppofed to be a circle, and it was thought proper briefly to demonftrate it; from whence it is fufficiently manifeft, that this plane divides the cylinder into two others: And the fame thing is underftood to be fupplied in prop. 14.

Z 4 P R O P.

P R O P. XV. B. XII.

" And complete the cylinders AX, EO," both the enuncia-
tion and expofition of the propofition reprefent the cylinders as
well as the cones, as already defcribed : Wherefore the read-
ing ought rather to be, " and let the cones be ALC, ENG ;
" and the cylinders AX, EO."

The firft cafe in the fecond part of the demonftration is want-
ing ; and fomething alfo in the fecond cafe of that part, before
the repetition of the conftruction is mentioned ; which are now
added.

P R O P. XVII. B. XII.

In the enunciation of this, propofition, the Greek words εἰς
την μείζονα σφαῖραν στερεον πολυεδρον ἐγγραψαι, ως ψαυον της
ἐλασσονος σφαιρας κατα την ἐπιφανειαν, are thus tranflated by Com-
mandine and others, " in majori folidum polyhedrum defcri-
" bere quod minoris fphaerae fuperficiem non tangat ;" that is,
" to defcribe in the greater fphere a folid polyhedron which
" fhall not meet the fuperficies of the leffer fphere :" Whereby
they refer the words κατα την ἐπιφανειαν to thefe next to them
της ἐλασσονος σφαιρας : But they ought by no means to be thus
tranflated, for the folid polyhedron doth not only meet the fu-
perficies of the leffer fphere, but pervades the whole of that
fphere : Therefore the forefaid words are to be referred to
το στερεον πολυεδρον, and ought thus to be tranflated, viz. to de-
fcribe in the greater fphere a folid polyhedron whofe fuperficies
fhall not meet the leffer fphere ; as the meaning of the propo-
fition neceffarily requires.

The demonftration of the propofition is fpoiled and mutila-
ted : For fome eafy things are very explicitly demonftrated,
while others not fo obvious are not fufficiently explained ; for
example, when it is affirmed, that the fquare of KB is greater
than the double of the fquare of BZ, in the firft demonftra-
tion ; and that the angle BZK is obtufe, in the fecond : Both
which ought to have been demonftrated : Befides, in the firft
demonftration, it is faid, " draw KΩ from the point K perpen-
" dicular to BD ;" whereas it ought to have been faid, " join
" KV," and it fhould have been demonftrated that KV is
perpendicular to BD : For it is evident from the figure in Her-
vagius's and Gregory's editions, and from the words of the
demon-

demonftration, that the Greek Editor did not perceive that
the perpendicular drawn from the point K to the ftraight line
BD muſt neceſſarily fall upon the point V, for in the figure it
is made to fall upon the point Ω a different point from V,
which is likewife fuppoſed in the demonftration. Comman-
dine feems to have been aware of this; for in this figure he
marks one and the fame point with the two letters V, Ω; and
before Commandine, the learned John Dee, in the commen-
tary he annexes to this propofition in Henry Billinſley's tranſ-
lation of the Elements printed at London, Ann. 1570, expreſsly
takes notice of this error, and gives a demonftration fuited to
the conftruction in the Greek text, by which he fhews that the
perpendicular drawn from the point K to BD, muſt neceſſarily
fall upon the point V.

Likewife it is not demonftrated that the quadrilateral figures
SOPT, TPRY, and the triangle YRX do not meet the leſſer
fphere, as was neceſſary to have been done: Only Clavius, as
far as I know, has obferved this, and demonftrated it by a lem-
ma, which is now premifed to this propofition, fomething alter-
ed and more briefly demonftrated.

In the corollary of this propofition, it is fuppoſed that a
folid polyhedron is defcribed in the other fphere fimilar to that
which is defcribed in the fphere BCDE; but, as the conftruction
by which this may be done is not given, it was thought proper
to give it, and to demonftrate, that the pyramids in it are fimi-
lar to thofe of the fame order in the folid polyhedron defcribed
in the fphere BCDE.

From the preceding notes, it is fufficiently evident how
much the Elements of Euclid, who was a moft accurate geo-
meter, have been vitiated and mutilated by ignorant editors.
The opinion which the greateſt part of learned men have en-
tertained concerning the prefent Greek edition, viz. that it is
very little or nothing different from the genuine work of Eu-
clid, has without doubt deceived them, and made them lefs
attentive and accurate in examining that edition; whereby fe-
veral errors, fome of them grofs enough, have efcaped their
notice from the age in which Theon lived to this time. Upon
which account there is fome ground to hope that the pains we
have taken in correcting thofe errors, and freeing the ele-
ments as far as we could from blemiſhes, will not be unac-
ceptable

Book XII. ceptable to good judges who can difcern when demonftrations are legitimate, and when they are not.

The objections which, fince the firft edition, have been made againft fome things in the notes, efpecially againft the doctrine of proportionals, have either been fully anfwered in Dr Barrow's Lect. Mathemat. and in thefe notes; or are fuch, except one which has been taken notice of in the note on prop. 1. book 11. as fhew that the perfon who made them has not fufficiently confidered the things againft which they are brought; fo that it is not neceffary to make any further anfwer to thefe objections and others like them againft Euclid's definition of proportionals; of which definition Dr Barrow juftly fays in page 297. of the above named book, that " Ni- " fi machinis impulfa validioribus aeternum perfiftet incon- " cuffa."

F　I　N　I　S.

EUCLID's DATA.

IN THIS EDITION

SEVERAL ERRORS ARE CORRECTED,

AND

SOME PROPOSITIONS ADDED.

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PREFACE.

EUCLID's DATA is the firſt in order of the books written by the antient geometers to facilitate and promote the method of reſolution or analyſis. In the general, a thing is ſaid to be given which is either actually exhibited, or can be found out, that is, which is either known by hypotheſis, or that can be demonſtrated to be known; and the propoſitions in the book of Euclid's Data ſhew what things can be found out or known from thoſe that by hypotheſis are already known; ſo that in the analyſis or inveſtigation of a problem, from the things that are laid down to be known or given, by the help of theſe propoſitions other things are demonſtrated to be given, and from theſe, other things are again ſhewn to be given, and ſo on, until that which was propoſed to be found out in the problem is demonſtrated to be given, and when this is done, the problem is ſolved, and its compoſition is made and derived from the compoſitions of the Data which were made uſe of in the analyſis. And thus the Data of Euclid are of the moſt general and neceſſary uſe in the ſolution of problems of every kind.

Euclid is reckoned to be the author of the Book of the Data, both by the antient and modern geometers; and there ſeems to be no doubt of his having written a book on this ſubject, but which, in the courſe of ſo many ages, has been much vitiated by unſkilful editors in ſeveral places, both in the order of the propoſitions, and in the definitions and demonſtrations themſelves. To correct

rect the errors which are now found in it, and bring it nearer to the accuracy with which it was, no doubt, at first written by Euclid, is the design of this edition, that so it may be rendered more useful to geometers, at least to beginners who desire to learn the investigatory method of the antients. And for their sakes, the compositions of most of the Data are subjoined to their demonstrations, that the compositions of problems solved by help of the Data may be the more easily made.

Marinus the philosopher's preface, which, in the Greek edition, is prefixed to the Data, is here left out, as being of no use to understand them. At the end of it, he says, that Euclid has not used the synthetical, but the analytical method in delivering them; in which he is quite mistaken; for, in the analysis of a theorem, the thing to be demonstrated is assumed in the analysis; but in the demonstrations of the Data, the thing to be demonstrated, which is, that something or other is given, is never once assumed in the demonstration, from which it is manifest, that every one of them is demonstrated synthetically; though, indeed, if a proposition of the Data be turned into a problem, for example the 84th or 85th in the former editions, which here are the 85th and 86th, the demonstration of the proposition becomes the analysis of the problem.

Wherein this edition differs from the Greek, and the reasons of the alterations from it, will be shewn in the notes at the end of the Data.

EUCLID's

E U C L I D's D A T A,

D E F I N I T I O N S.

I.

SPACES, lines, and angles, are faid to be given in magni‑
tude, when equals to them can be found.

II.

A ratio is faid to be given, when a ratio of a given magnitude
to a given magnitude which is the fame ratio with it can be
found.

III.

Rectilineal figures are faid to be given in fpecies, which have
each of their angles given, and the ratios of their fides given.

IV.

Points, lines, and fpaces, are faid to be given in pofition, which
have always the fame fituation, and which are either actually
exhibited, or can be found.

A.

An angle is faid to be given in pofition, which is contained by
ftraight lines given in pofition.

V.

A circle is faid to be given in magnitude, when a ftraight line
from its centre to the circumference is given in magnitude.

VI.

A circle is faid to be given in pofition and magnitude, the
centre of which is given in pofition, and a ftraight line from
it to the circumference is given in magnitude.

VII.

Segments of circles are faid to be given in magnitude, when
the angles in them, and their bafes, are given in magnitude.

VIII.

Segments of circles are faid to be given in pofition and mag‑
nitude, when the angles in them are given in magnitude,
and their bafes are given both in pofition and magnitude.

IX.

A magnitude is faid to be greater than another by a given
magnitude, when this given magnitude being taken from it,
the remainder is equal to the other magnitude.

X.

X.

A magnitude is said to be lefs than another by a given magni-
tude, when this given magnitude being added to it, the
whole is equal to the other magnitude.

PROPOSITION I.

THE ratios of given magnitudes to one another is
given.

Let A, B be two given magnitudes, the ratio of A to B is
given.

Becaufe A is a given magnitude, there may
ᵃ be found one equal to it; let this be C : And
becaufe B is given, one equal to it may be
found ; let it be D ; and fince A is equal to C,
and B to D ; therefore ᵇ A is to B, as C to
D ; and confequently the ratio of A to B is
given, becaufe the ratio of the given magni-
tudes C, D which is the fame with it, has been **A B C D**
found.

PROP. II.

IF a given magnitude has a given ratio to another
magnitude, " and if unto the two magnitudes by
" which the given ratio is exhibited, and the given
" magnitude, a fourth proportional can be found ;" the
other magnitude is given.

Let the given magnitude A have a given ratio to the mag-
nitude B; if a fourth proportional can be found to the three
magnitudes above named, B is given in magnitude.

Becaufe A is given, a magnitude may be
found equal to it ᵃ ; let this be C ; And be-
caufe the ratio of A to B is given, a ratio
which is the fame with it may be found, let
this be the ratio of the given magnitude E **A B C D**
to the given magnitude F : Unto the magni-   **E F**
tudes E, F, C find a fourth proportional
D, which, by the hypothefis, can be done.
Wherefore, becaufe A is to B, as E to F; and
as E to F, fo is C to D ; A is ᵇ to B, as C to
                                                          D.
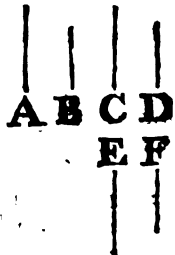
D. But A is equal to C ; therefore ^c B is equal to D. The c 14. 5.
magnitude B is therefore given ^a, becaufe a magnitude D equal a 1. def.
to it has been found.

The limitation within the inverted commas is not in the
Greek text, but is now neceffarily added ; and the fame muft be
underftood in all the propofitions of the book which depend
upon this fecond propofition, where it is not exprefsly mention-
ed. See the note upon it.

P R O P. III. 3.

IF any given magnitudes be added together, their fum
fhall be given.

Let any given magnitudes AB, BC be added together, their
fum AC is given.

Becaufe AB is given, a magnitude equal to it may be found ^a ; a 1. def.
let this be DE : And becaufe BC is gi-
ven, one equal to it may be found ; let
this be EF : Wherefore, becaufe AB is
equal to DE, and BC equal to EF ; the
whole AC is equal to the whole DF :
AC is therefore given, becaufe DF has been found which is e-
qual to it.

P R O P. IV. 4.

IF a given magnitude be taken from a given magni-
tude ; the remaining magnitude fhall be given.

From the given magnitude AB, let the given magnitude AC
be taken ; the remaining magnitude CB is given.

Becaufe AB is given, a magnitude equal to it may ^a be a 1. def.
found; let this be DE : And becaufe
AC is given, one equal to it may be
found ; let this be DF : Wherefore, be-
caufe AB is equal to DE, and AC to
DF ; the remainder CB is equal to the
remainder FE. CB is therefore given ^a, becaufe FE which is
equal to it has been found.

A a P R O P.

12

See N.

IF of three magnitudes, the firſt together with the ſecond be given, and alſo the ſecond together with the
third ; either the firſt is equal to the third, or one of them
is greater than the other by a given magnitude.

Let AB, BC, CD be three magnitudes, of which AB together
with BC, that is AC, is given ; and alſo BC together with CD,
that is BD, is given.   Either AB is equal to CD, or one of them
is greater than the other by a given magnitude.
Becauſe AC, BD are each of them given, they are either equal to one another, or not equal.
Firſt, let them be equal, and becauſe
AC is equal to BD, take away the common part BC ; therefore
the remainder AB is equal to the remainder CD.
But if they be unequal, let AC be greater than BD, and
make CE equal to BD.   Therefore CE is given, becauſe BD is
given.   And the whole AC is gi-

a 4. dat.   ven ; therefore ª AE the remainder
is given.   And becauſe EC is equal
to BD, by taking BC from both,
the remainder EB is equal to the remainder CD.   And AE is
given ; wherefore AB exceeds EB, that is CD, by the given
magnitude AE.

5.

See N.

IF a magnitude has a given ratio to a part of it, it ſhall
alſo have a given ratio to the remaining part of it.

Let the magnitude AB have a given ratio to AC a part of it ;
it has alſo a given ratio to the remainder BC.
Becauſe the ratio of AB to AC is given, a ratio may be

a 2. def.   found ª which is the ſame to it : Let this be the ratio of DE
a given magnitude to the given mag-
nitude DF.   And becauſe DE, DF are

b 4. dat.   given, the remainder FE is ᵇ given :
And becauſe AB is to AC, as DE to

c E. 5.   DF, by converſion ᶜ AB is to BC, as
DE to EF.   Therefore the ratio of AB to BC is given, becauſe
the ratio of the given magnitudes DE, EF, which is the ſame
with it, has been found.

                                                COR.

COR. From this it follows, that the parts AC, CB have a given ratio to one another: Because as AB to BC, so is DE to EF; by division [d], AC is to CB, as DF to FE; and DF, FE are given; therefore [a] the ratio of AC to CB is given. d 17. 5. a 2. def.

PROP. VII.   6.

IF two magnitudes which have a given ratio to one another, be added together; the whole magnitude shall have to each of them a given ratio.   See N.

Let the magnitudes AB, BC which have a given ratio to one another, be added together; the whole AC has to each of the magnitudes AB, BC a given ratio.

Because the ratio of AB to BC is given, a ratio may be found [a] which is the same with it; let this be the ratio of the given magnitudes DE, EF: And because DE, EF are given, the whole DF is given [b]: And because as AB to BC, so is DE to EF; by composition [c] AC is to CB, as DF to FE; and by conversion [d], AC is to AB, as DF to DE: Wherefore because AC is to each of the magnitudes AB, BC, as DF to each of the others DE, EF; the ratio of AC to each of the magnitudes AB, BC is given [a].   a 2. def. b 3. dat. c 18. 5. d E. 5.

PROP. VIII.   7.

IF a given magnitude be divided into two parts which have a given ratio to one another, and if a fourth proportional can be found to the sum of the two magnitudes by which the given ratio is exhibited, one of them, and the given magnitude; each of the parts is given.   See N.

Let the given magnitude AB be divided into the parts AC, CB which have a given ratio to one another; if a fourth proportional can be found to the above named magnitudes; AC and CB are each of them given.

Because the ratio of AC to CB is given, the ratio of AB to BC is given [a]; therefore a ratio;   a 7. dat.

A a 2   which

which is the fame with it can be found ^b, let this be the ratio of
the given magnitudes DE, EF : And
becaufe the given magnitude AB has
to BC the given ratio of DE to EF, if
unto DE, EF, AB a fourth propor-
tional can be found, this which is BC
is given ^c; and becaufe AB is given, the other part AC is
given ^d.
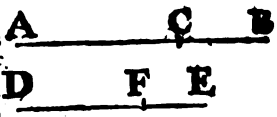
c 2. dat.
d 4. dat.

In the fame manner, and with the like limitation, if the dif-
ference AC of two magnitudes AB, BC which have a given
ratio be given ; each of the magnitudes AB, BC is given.

PROP. IX.

MAGNITUDES which have given ratios to the
fame magnitude, have alfo a given ratio to one
another.

Let A, C have each of them a given ratio to B; A has a gi-
ven ratio to C.

Becaufe the ratio of A to B is given, a ratio which is the
a 1. def. fame to it may be found ^a; let this be the ratio of the given
magnitudes D, E : And becaufe the ratio of B to C is given, a
ratio which is the fame with it may be found ^a; let this be the
ratio of the given magnitudes F, G:
To F, G, E find a fourth propor-
tional H, if it can be done; and
becaufe as A is to B, fo is D to E;
and as B to C, fo is (F to G, and
fo is) E to H; ex aequali, as A to
C, fo is D to H : Therefore the ratio
of A to C is given ^a, becaufe the
ratio of the given magnitudes D and
H, which is the fame with it, has
been found : But if a fourth propor-
tional to F, G, E cannot be found,
then it can only be faid that the ratio of A to C is compounded
of the ratios of A to B, and B to C, that is, of the given ratios
of D to E, and F to G.
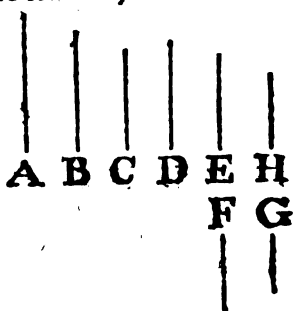
PROP. X.

IF two or more magnitudes have given ratios to one another, and if they have given ratios, though they be not the same, to some other magnitudes; thefe other magnitudes fhall alfo have given ratios to one another.

Let two or more magnitudes A, B, C have given ratios to one another; and let them have given ratios, though they be not the fame, to fome other magnitudes D, E, F: The magnitudes D, E, F have given ratios to one another.

Becaufe the ratio of A to B is given, and likewife the ratio of A to D; therefore the ratio of D to B is given ᵃ; but the ratio of B to E is given, therefore ᵃ the ratio of D to E is given : And becaufe the ratio of B to C is given, and alfo the ratio of B to E; the ratio of E to C is given ᵃ : And the ratio of C to F is given; wherefore the ratio of E to F is given: D, E, F have therefore given ratios to one another.

A———————D————————
B————— E————————
C————— F—————

ᵃ). dat.
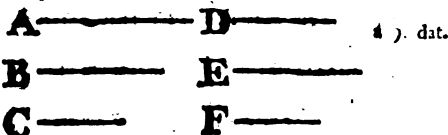
PROP. XI.

IF two magnitudes have each of them a given ratio to another magnitude, both of them together fhall have a given ratio to that other.

Let the magnitudes AB, BC have a given ratio to the magnitude D; AC has a given ratio to the fame D.

Becaufe AB, BC have each of them a given ratio to D, the ratio of AB to BC is given ᵃ : And by compofition, the ratio of AC to CB is given ᵇ : But the ratio of BC to D is given; therefore ᵃ the ratio of AC to D is given.

A   B   C
—————————————
D
————

a 9. dat.

b 7. dat.

A a 3   PROP.

23.
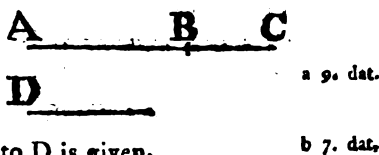
PROP. XII.

See N. IF the whole have to the whole a given ratio, and the parts have to the parts given, but not the same, ratios: Every one of them, whole or part, ſhall have to every one a given ratio.

Let the whole AB have a given ratio to the whole CD, and the parts AE, EB have given, but not the same, ratios to the parts CF, FD : Every one ſhall have to every one, whole or part, a given ratio.

Becauſe the ratio of AE to CF is given, as AE to CF, ſo make AB to CG ; the ratio therefore of AB to CG is given ; wherefore the ratio of the remainder EB to the remainder FG is given, becauſe it is the ſame [a] with the ratio of AB to CG : And the ratio of EB to FD is given, wherefore the ratio of FD to FG is given [b] ; and by converſion, the ratio of FD to DG is given [c] : And becauſe AB has to each of the magnitudes CD, CG a given ratio, the ratio of CD to CG is given [b] ; and therefore [c] the ratio of CD to DG is given : But the ratio of GD to DF is given, wherefore [b] the ratio of CD to DF is given, and conſequently [d] the ratio of CF to FD is given ; but the ratio of CF to AE is given, as alſo the ratio of FD to EB ; wherefore [e] the ratio of AE to EB is given ; as alſo the ratio of AB to each of them [f] : The ratio therefore of every one to every one is given.

[a] 19. 5.
[b] 9. dat.
[c] 6. dat.
[d] cor. 6. dat.
[e] 10. dat.
[f] 7. dat.

24.

PROP. XIII.

See N. IF the firſt of three proportional ſtraight lines has a given ratio to the third, the firſt ſhall alſo have a given ratio to the ſecond.

Let A, B, C be three proportional ſtraight lines, that is, as A to B, ſo is B to C ; if A has to C a given ratio, A ſhall alſo have to B a given ratio.

Becauſe the ratio of A to C is given, a ratio which is the ſame with it may be found [a] ; let this be the ratio of the given ſtraight lines D, E ; and between D and E find a [b] mean

[a] 2. def.
[b] 13. 6.

proportional

proportional F; therefore the rectangle contained by D and
E is equal to the square of F, and the rect-
angle D, E is given, becaufe its fides D, E are
given; wherefore the fquare of F, and the
ftraight line F is given : And becaufe as A is
to C, fo is D to E; but as A to C, fo is [c] the
fquare of A to the fquare of B; and as D to
E, fo is [c] the fquare of D to the fquare of F:
therefore the fquare [d] of A is to the fquare of
B, as the fquare of D to the fquare of F:
As therefore [e] the ftraight line A to the
ftraight line B, fo is the ftraight line D to the
ftraight line F: Therefore the ratio of A to B
is given [a], becaufe the ratio of the given
ftraight lines D, F which is the fame with it
has been found.

c 2. cor.
20. 6.

d 11. 5.

e 22. 6.

a 2. def.
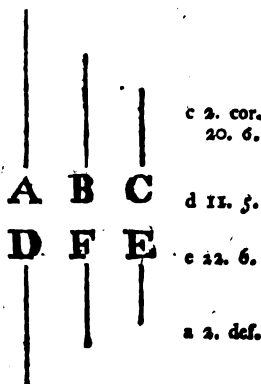
PROP. XIV.

A.

IF a magnitude together with a given magnitude has a
given ratio to another magnitude; the excefs of this
other magnitude above a given magnitude has a given ra-
tio to the firft magnitude: And if the excefs of a magni-
tude above a given magnitude has a given ratio to ano-
ther magnitude; this other magnitude together with a
given magnitude has a given ratio to the firft magnitude.

Let the magnitude AB together with the given magnitude
BE, that is AE, have a given ratio to the magnitude CD ; the
excefs of CD above a given magnitude has a given ratio to AB.
Becaufe the ratio of AE to CD is given, as AE to CD, fo
make BE to FD; therefore the ratio of BE to FD is given, and
BE is given; wherefore FD is given [a]:
And becaufe as AE to CD, fo is BE
to FD, the remainder AB is [b] to the
remainder CF, as AE to CD: But the
ratio of AE to CD is given, therefore
the ratio of AB to CF is given; that is, CF the excefs of CD
above the given magnitude FD has a given ratio to AB.
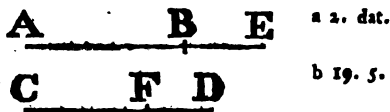
a 2. dat.

b 19. 5.

Next, Let the excefs of the magnitude AB above the given
magnitude BE, that is, let AE have a given ratio to the mag-

A a 4   nitude

nitude CD: CD together with a given magnitude has a given ratio to AB.

Becaufe the ratio of AE to CD is given, as AE to CD, fo make BE to FD; therefore the ratio of BE to FD is given, and BE is given,

a 2. dat. whrefore FD is given [a]: And becaufe as AE to CD, fo is BE to FD, AB is

c 12. 5. to CF, as [c] AE to CD: But the ratio of AE to CD is given, therefore the ratio of AB to CF is given; that is, CF which is equal to CD together with the given magnitude DF has a given ratio to AB.

E.

P R O P. XV.

See N. IF a magnitude together with that to which another magnitude has a given ratio, be given; the fum of this other, and that to which the firft magnitude has a given ratio is given.

Let AB, CD be two magnitudes of which AB together with BE to which CD has a given ratio, is given; CD is given together with that magnitude to which AB has a given ratio.

Becaufe the ratio of CD to BE is given, as BE to CD, fo make AE to FD; therefore the ratio of AE to FD is given,

x 2. dat. and AE is given, wherefore [x] FD is given: And becaufe as BE to

b Cor. 19. CD, fo is AE to FD: AB is [b] to
5. FC, as BE to CD: And the ratio of BE to CD is given, wherefore the ratio of AB to FC is given: And FD is given, that is CD together with FC to which AB has a given ratio is given.

10.

P R O P. XVI.

See N. IF the excefs of a magnitude above a given magnitude, has a given ratio to another magnitude; the excefs of both together above a given magnitude fhall have to that other a given ratio: And if the excefs of two magnitudes together above a given magnitude, has to one of them a given ratio; either the excefs of the other above a given magnitude has to that one a given ratio; or the other is given together with the magnitude to which that one has a given ratio.

Let

Let the excefs of the magnitude AB above a given magnitude, have a given ratio to the magnitude BC; the excefs of AC, both of them together, above the given magnitude, has a given ratio to BC.

Let AD be the given magnitude, the excefs of AB above which, viz. DB has a given ratio to BC: And becaufe DB has a given ratio to BC, the ratio of DC to CB is given [a], and AD is given; therefore DC, the excefs of AC above the given magnitude AD, has a given ratio to BC. [a 7. dat.

A   D   B   C

Next, let the excefs of two magnitudes AB, BC together, above a given magnitude, have to one of them BC a given ratio; either the excefs of the other of them AB above the given magnitude fhall have to BC a given ratio; or AB is given, together with the magnitude to which BC has a given ratio.

A   D   B   E C

Let AD be the given magnitude, and firft let it be lefs than AB; and becaufe DC the excefs of AC above AD has a given ratio to BC, DB has [b] a given ratio to BC; that is, DB, the excefs of AB above the given magnitude AD, has a given ratio to BC. [b Cor. 6. dat.

But let the given magnitude be greater than AB, and make AE equal to it; and becaufe EC, the excefs of AC above AE, has to BC a given ratio, BC has [c] a given ratio to BE; and becaufe AE is given, AB together with BE to which BC has a given ratio, is given. [c 6. dat.

P R O P. XVII.

[II.]

IF the excefs of a magnitude above a given magnitude has a given ratio to another magnitude; the excefs of the fame firft magnitude above [a given magnitude, fhall have a given ratio to both the magnitudes together. And if the excefs of either of two magnitudes above a given magnitude has a given ratio to both magnitudes together; the excefs of the fame above a given magnitude fhall have a given ratio to the other.

Let the excefs of the magnitude AB above a given magnitude have a given ratio to the magnitude BC; the excefs of AB above a given magnitude has a given ratio to AC.

Let

Let AD be the given magnitude; and becaufe DB, the ex-
cefs of AB above AD, has a given ratio to BC; the ratio of
DC to DB is given ᵉ: Make the ratio of AD to DE the fame
with this ratio; therefore the ratio
of AD to DE is given: And AD

A _____ E _ D B ____ C

is given, wherefore ᵇ DE, and the remainder AE are given: And
becaufe as DC to DB, fo is AD to DE, AC is ᶜ to EB, as DC
to DB; and the ratio of DC to DB is given; wherefore the
ratio of AC to EB is given: And becaufe the ratio of EB to AC
is given, and that AE is given, therefore EB the excefs of AB
above the given magnitude AE, has a given ratio to AC.

Next, let the excefs of AB above a given magnitude have a
given ratio to AB and BC together, that is, to AC; the excefs
of AB above a given magnitude has a given ratio to BC.

Let AE be the given magnitude; and becaufe EB the excefs
of AB above AE has to AC a given ratio, as AC to EB, fo
make AD to DE; therefore the ratio of AD to DE is given,
as alfo ᵈ the ratio of AD to AE: And AE is given, where-
fore ᵇ AD is given: And becaufe, as the whole AC, to the
whole EB, fo is AD to DE; the remainder DC is ᵉ to the
remainder DB, as AC to EB; and the ratio of AC to EB is
given; wherefore the ratio of DC to DB is given, as alfo ᶠ the
ratio of DB to BC: And AD is given; therefore DB, the ex-
cefs of AB above a given magnitude AD, has a given ratio to
BC.

a 7. dat.
b 2. dat.
c 12. 5.
d 6. dat.
e 19. 5.
f Cor. 6. dat.

PROP. XVIII.

14.

IF to each of two magnitudes, which have a given ratio
to one another, a given magnitude be added; the
whole fhall either have a given ratio to one another, or
the excefs of one of them above a given magnitude fhall
have a given ratio to the other.

Let the two magnitudes AB, CD have a given ratio to one
another, and to AB let the given magnitude BE be added, and
the given magnitude DF to CD: The wholes AE, CF either
have a given ratio to one another, or the excefs of one of them
above a given magnitude has a given ratio to the other ᵃ.
Becaufe BE, DF are each of them given, their ratio is given,
                                                          and

a 1. dat.

and if this ratio be the fame with
the ratio of AB to CD, the ratio of
AE to CF, which is the fame [b] with
the given ratio of AB to CD, fhall
be given.

But if the ratio of BE to DF be not the fame with the ratio
of AB to CD, either it is greater than the ratio of AB to
CD, or, by inverfion, the ratio of DF to BE is greater than
the ratio of CD to AB: Firft, let
the ratio of BE to DF be greater
than the ratio of AB to CD; and as
AB to CD, fo make BG to DF;
therefore the ratio of BG to DF is
given; and DF is given, therefore [c] BG is given: And becaufe
BE has a greater ratio to DF than (AB to CD, that is, than)
BG to DF, BE is greater [d] than BG: And becaufe as AB to
CD, fo is BG to DF; therefore AG is [b] to CF, as AB to CD:
But the ratio of AB to CD is given, wherefore the ratio of AG
to CF is given; and becaufe BE, BG are each of them given,
GE is given: Therefore AG, the excefs of AE above a given
magnitude GE, has a given ratio to CF. The other cafe is de-
monftrated in the fame manner.
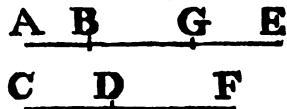
P R O P. XIX.

IF from each of two magnitudes, which have a given
ratio to one another, a given magnitude be taken, the
remainders fhall either have a given ratio to one another,
or the excefs of one of them above a given magnitude,
fhall have a given ratio to the other.

Let the magnitudes AB, CD have a given ratio to one an-
other, and from AB let the given magnitude AE be taken,
and from CD, the given magnitude CF: The remainders EB,
FD fhall either have a given ratio to one another, or the ex-
cefs of one of them above a gi-
ven magnitude fhall have a given
ratio to the other.
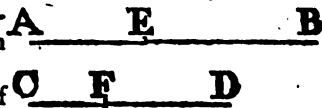
Becaufe. AE, CF are each of
them given, their ratio is given [a];
and if this ratio be the fame with the ratio of AB to CD, the

ratio

b. 19. 5.

ratio of the remainder EB to the remainder FD, which is the same^b with the given ratio of AB to CD, shall be given.

But if the ratio of AB to CD be not the same with the ratio of AE to CF, either it is greater than the ratio of AE to CF; or, by inversion, the ratio CD to AB is greater than the ratio of CF to AE : First, let the ratio of AB to CD be greater than the ratio of AE to CF, and as AB to CD, so make AG to CF; therefore the ratio of AG to CF is given, and CF is given,

c 2. dat.

wherefore^c AG is given : And because the ratio of AB to CD, that is, the ratio of AG to CF,

A   E G   B

C   F   D
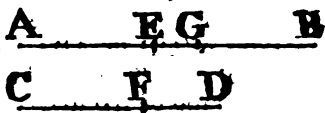
d 10. 5.

is greater than the ratio of AE to CF; AG is greater ^d than AE : And AG, AE, are given, therefore the remainder EG is given ; and as AB to CD, so is AG to CF, and so is ^b the remainder GB to the remainder FD ; and the ratio of AB to CD is given : Wherefore the ratio of GB to FD is given ; therefore GB, the excess of EB above a given magnitude EG, has a given ratio to FD. In the same manner the other case is demonstrated.

16.

P R O P. XX.

IF to one of two magnitudes which have a given ratio to one another, a given magnitude be added, and from the other a given magnitude be taken ; the excess of the sum above a given magnitude shall have a given ratio to the remainder.

Let the two magnitudes AB, CD have a given ratio to one another, and to AB let the given magnitude EA be added, and from CD let the given magnitude CF be taken ; the excess of the sum EB above a given magnitude has a given ratio to the remainder FD.

Because the ratio of AB to CD is given, make as AB to CD, so AG to CF: Therefore the ratio of AG to CF is given,

a 2. dat.

and CF is given, wherefore^a AG is given ; and EA is given, therefore the whole EG is given : And because as AB to CD, so is AG

E   A   G   B

C   F   D

b 19. 5.

to CF, and so is ^b the remainder GB to the remainder FD ; the ratio of GB to FD is given. And EG is given, therefore GB, the excess of the sum EB above

bove the given magnitude EG, has a given ratio to the remainder FD.

PROP. XXI.

C.

IF two magnitudes have a given ratio to one another, See N. if a given magnitude be added to one of them, and the other be taken from a given magnitude; the sum, together with the magnitude to which the remainder has a given ratio, is given: And the remainder is given together with the magnitude to which the sum has a given ratio.

Let the two magnitudes AB, CD have a given ratio to one another; and to AB let the given magnitude BE be added, and let CD be taken from the given magnitude FD: The sum AE is given together with the magnitude to which the remainder FC has a given ratio.

Because the ratio of AB to CD is given, make as AB to CD, so GB to FD: Therefore the ratio of GB to FD is given, and FD is given, wherefore GB is given[a]; and BE is given, the whole GE is therefore given: And because as AB to CD, so is GB to FD, and so is[b] GA to FC; the

a 2. dat,

b 19. 5.

ratio of GA to FC is given: And AE together with GA is given, because GE is given; therefore the sum AE together with GA, to which the remainder FC has a given ratio, is given. The second part is manifest from prop. 15.
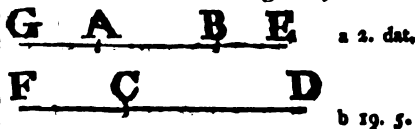
PROP. XXII.

D.

IF two magnitudes have a given ratio to one another, if See N. from one of them a given magnitude be taken, and the other be taken from a given magnitude; each of the remainders is given together with the magnitude to which the other remainder has a given ratio.

Let the two magnitudes AB, CD have a given ratio to one another, and from AB let the given magnitude AE be taken,

and

and let CD be taken from the given magnitude CF; the remainder EB is given together with the magnitude to which the other remainder DF has a given ratio.

Becaufe the ratio of AB to CD is given, make as AB to CD, fo AG to CF: The ratio of AG to CF is therefore given, and CF is given, wherefore [a] AG is given; and AE is given, and therefore the remainder EG is given: And becaufe as AB to CD, fo is AG to CF: And fo is [b] the remainder BG to the remainder DF; the ratio of BG to DF is given: And EB together with BG is given, becaufe EG is given: Therefore the remainder EB together with BG, to which DF the other remainder has a given ratio, is given. The fecond part is plain from this and prop. 15.

A   E B   G

C   D F
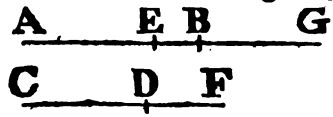
P R O P. XXIII.

IF, from two given magnitudes there be taken magnitudes which have a given ratio to one another, the remainders fhall either have a given ratio to one another, or the excefs of one of them above a given magnitude fhall have a given ratio to the other.

Let AB, CD be two given magnitudes, and from them let the magnitudes AE, CF, which have a given ratio to one another, be taken; the remainders EB FD either have a given ratio to one another, or the excefs of one of them above a given magnitude has a given ratio to the other.

Becaufe AB, CD are each of them given, the ratio of AB to CD is given: And if this ratio be the fame with the ratio of AE to CF, then the remainder EB has [a] the fame given ratio to the remainder FD.
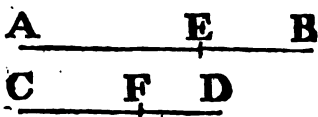
A   E   B

C   F   D

But if the ratio of AB to CD be not the fame with the ratio of AE to CF, it is either greater than it, or, by inverfion, the ratio of CD to AB is greater than the ratio of CF to AE: Firft, let the ratio of AB to CD be greater than the ratio of AE to CF; and as AE to CF, fo make AG to CD; therefore the ratio of AG to CD is given, becaufe the ratio of AE to CF is given; and CD is given, wherefore [b] AG is given;

given;

given; and becaufe the ratio of AB to CD is greater than the ratio of (AE to CF, that is, than the ratio of) AG to CD; AB is greater[c] than AG: And AB, AG are given; therefore the remainder BG is given: And becaufe as AE to CF, fo is AG to CD, and fo is[a] EG to FD; the ratio of EG to FD is given: And GB is given; therefore EG, the excefs of EB above a given magnitude GB, has a given ratio to FD. The other cafe is fhewn in the fame way.

A   E   G B

C   F   D

c 10. 5.

a 19. 5.

P R O P. XXIV.

13.

IF there be three magnitudes, the firft of which has a given ratio to the fecond, and the excefs of the fecond above a given magnitude has a given ratio to the third; the excefs of the firft above a given magnitude fhall alfo have a given ratio to the third.

See N.

Let AB, CD, E, be the three magnitudes of which AB has a given ratio to CD; and the excefs of CD above a given magnitude has a given ratio to E: The excefs of AB above a given magnitude has a given ratio to E.

Let CF be the given magnitude, the excefs of CD above which, viz. FD, has a given ratio to E: And becaufe the ratio of AB to CD is given, as AB to CD, fo make AG to CF; therefore the ratio of AG to CF is given; and CF is given, wherefore[a] AG is given: And becaufe as AB to CD, fo is AG to CF, and fo is[b] GB to FD; the ratio of GB to FD is given. And the ratio of FD to E is given, wherefore[c] the ratio of GB to E is given, and AG is given; therefore GB the excefs of AB above a given magnitude AG has a given ratio to E.

a 2. dat.

b 19. 5.

c 9. dat.

A C F G B D E

COR. 1. And if the firft has a given ratio to the fecond, and the excefs of the firft above a given magnitude has a given ratio to the third; the excefs of the fecond above a given magnitude fhall have a given ratio to the third. For, if the fecond be called the firft, and the firft the fecond, this corollary will be the fame with the propofition.

COR.
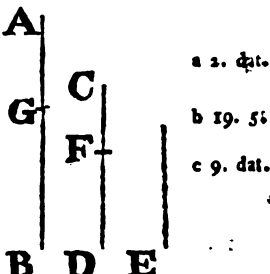
Cor. 2. Also, if the first has a given ratio to the second, and the excess of the third above a given magnitude has also a given ratio to the second, the same excess shall have a given ratio to the first; as is evident from the 9th dat.

27.

P R O P. XXV.

IF there be three magnitudes, the excess of the first whereof above a given magnitude has a given ratio to the second; and the excess of the third above a given magnitude has a given ratio to the same second : The first shall either have a given ratio to the third, or the excess of one of them above a given magnitude shall have a given ratio to the other.

Let AB, C, DE be three magnitudes, and let the excesses of each of the two AB, DE above given magnitudes have given ratios to C; AB, DE either have a given ratio to one another, or the excess of one of them above a given magnitude has a given ratio to the other.

Let FB the excess of AB above the given magnitude AF have a given ratio to C; and let GE the excess of DE above the given magnitude DG have a given ratio to C; and because FB, GE have each of them a given ratio to C, they

a 9. dat.   have a given ratio[a] to one another. But to FB, GE the given magnitudes AF, DG are add-

b 18. dat. ed; therefore[b] the whole magnitudes AB, DE have either a given ratio to one another, or the excess of one of them above a given magnitude has a given ratio to the other.
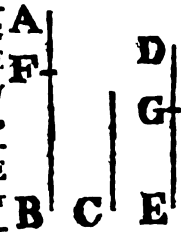
28.

P R O P. XXVI.

IF there be three magnitudes, the excesses of one of which above given magnitudes have given ratios to the other two magnitudes; these two shall either have a given ratio to one another, or the excess of one of them above a given magnitude shall have a given ratio to the other.

Let

Let AB, CD, EF be three magnitudes, and let GD the excefs of one of them CD above the given magnitude CG have a given ratio to AB; and alfo let KD the excefs of the fame CD above the given magnitude CK have a given ratio to EF, either AB has a given ratio to EF, or the excefs of one of them above a given magnitude has a given ratio to the other.

Becaufe GD has a given ratio to AB, as GD to AB, fo make CG to HA; therefore the ratio of CG to HA is given; and CG is given, wherefore [a] HA is given: And becaufe as GD to AB, fo is CG to HA, and fo is [b] CD to HB; the ratio of CD to HB is given: Alfo becaufe KD has a given ratio to EF, as KD to EF, fo make CK to LE; therefore the ratio of CK to LE is given; and CK is given, wherefore LE [a] is given: And becaufe as KD to EF, fo is CK to LE, and fo [b] is CD to LF; the raito of CD to LF is given: But the ratio of CD to HB is given, wherefore [c] the ratio of HB to LF is given: and from HB, LF the given magnitudes HA, LE being taken, the remainders AB, EF fhall either have a given ratio to one another, or the excefs of one of them above a given magnitude has a given ratio to the other [d].

a 2. dat.

b 12. 5.

c 9. dat.

d 19. dat.
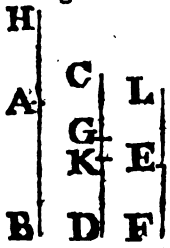
" Another Demonftration.

Let AB, C, DE be three magnitudes, and let the exceffes of one of them C above given magnitudes have given ratios to AB and DE; either AB, DE have a given ratio to one another, or the excefs of one of them above a given magnitude has a given ratio to the other.

Becaufe the excefs of C above a given magnitude has a given ratio to AB; therefore [a] AB together with a given magnitude has a given ratio to C: Let this given magnitude be AF, wherefore FB has a given ratio to C: Alfo, becaufe the excefs of C above a given magnitude has a given ratio to DE; therefore [a] DE together with a given magnitude has a given ratio to C: Let this given magnitude be DG, wherefore GE has a given ratio to C; And FB has a given ratio to C, therefore [b] the ratio of FB to GE is given: And from FB, GE the given magnitudes AF, DG being taken, the remainders AB, DE either have a given ratio to one another, or the excefs of one of them above a given magnitude has a given ratio to the other [e]."

a 14. dat.

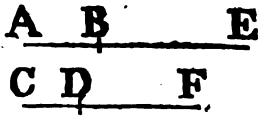b 9. dat.

e 19. dat.

B b

P R O P.

P R O P. XXVII.

IF there be three magnitudes: The excefs of the firft of which above a given magnitude has a given ratio to the fecond; and the excefs of the fecond above a given magnitude has alfo a given ratio to the third: The excefs of the firft above a given magnitude fhall have a given ratio to the third.

Let AB, CD, E be three magnitudes, the excefs of the firft of which AB above the given magnitude AG, viz. GB, has a given ratio to CD; and FD the excefs of CD above the given magnitude CF, has a given ratio to E: The excefs of AB above a given magnitude has a given ratio to E.

Becaufe the ratio of GB to CD is given, as GB to CD, fo make GH to CF; therefore the ratio of GH to CF is given; and CF is given, wherefore [a] GH is given; and AG is given, wherefore the whole AH is given: And becaufe as GB to CD, fo is GH to CF, and fo is [b] the remainder HB to the remainder FD; the ratio of HB to FD is given: And the ratio of FD to E is given, wherefore [c] the ratio of HB to E is given: And AH is given; therefore HB the excefs of AB above a given magnitude AH has a given ratio to E.

a 2. dat.

b 19. 5.

c 9. dat.
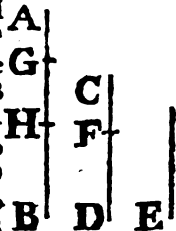
" *Otherwife,*

Let AB, C, D be three magnitudes, the excefs EB of the firft of which AB above the given magnitude AE has a given ratio to C, and the excefs of C above a given magnitude has a given ratio to D: The excefs of AB above a given magnitude has a given ratio to D.

Becaufe EB has a given ratio to C, and the excefs of C above a given magnitude has a given ratio to D; therefore [d] the excefs of EB above a given magnitude has a given ratio to D: Let this given magnitude be EF; therefore FB the excefs of EB above EF has a given ratio to D: And AF is given, becaufe AE, EF

d 24. dat.
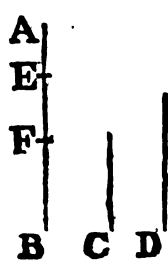
are

are given : Therefore FB the excefs of AB above a given magnitude AF has a given ratio to D."

P R O P. XXVIII.

25.

IF two lines given in pofition cut one another, the point See N.
or points in which they cut one another are given.

Let two lines AB, CD given in pofition cut one another in
the point E; the point E is given.

Becaufe the lines AB, CD
are given in pofition, they have
always the fame fituation [a], and
therefore the point, or points
in which they cut one another
have always the fame fituation :
And becaufe the lines AB, CD
can be found [a], the point, or
points, in which they cut one
another, are likewife found ;
and therefore are given in pofition [a].
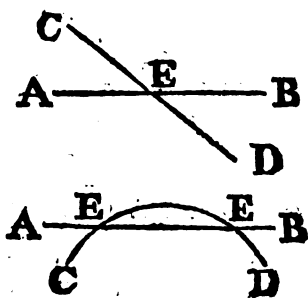
a 4. def.

P R O P. XXIX.

26.

IF the extremities of a ftraight line be given in pofition ; the ftraight line is given in pofition and magnitude.

Becaufe the extremities of the ftraight line are given, they
can be found [a] : Let thefe be the points A, B, between which a 4. def.
a ftraight line AB can be drawn [b];   A——————B   b 1. Poftulate.
this has an invariable pofition, becaufe between two given points there
can be drawn but one ftraight line : And when the ftraight line
AB is drawn, its magnitude is at the fame time exhibited, or
given : Therefore the ftraight line AB is given in pofition and
magnitude.
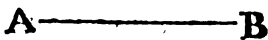
                        P R O P.

P R O P. XXX.

IF one of the extremities of a ftraight line given in po-
fition and magnitude be given; the other extremity
fhall alfo be given.

Let the point A be given, to wit, one of the extremities of
a ftraight line given in magnitude, and which lies in the ftraight
line AC given in pofition; the other extremity is alfo given.

Becaufe the ftraight line is given in magnitude, one equal
to it can be found ᵃ; let this be the ftraight line D: From the
greater ftraight line AC cut off AB
equal to the leffer D: Therefore the
other extremity B of the ftraight line
AB is found: And the point B has al-
ways the fame fituation; becaufe any
other point in AC, upon the fame fide of A, cuts off between
it and the point A a greater or lefs ftraight line than AB, that is,
than D: Therefore the point B is given ᵇ: And it is plain an-
other fuch point can be found in AC produced upon the other
fide of the point A.

a 1. def.

b 4. def.

P R O P. XXXI.

IF a ftraight line be drawn through a given point parallel
to a ftraight line given in pofition; that ftraight line
is given in pofition.

Let A be a given point, and BC a ftraight line given in pofi-
tion; the ftraight line drawn through A parallel to BC is given
in pofition.

Through A draw ᵃ the ftraight line
DAE parallel to BC; the ftraight
line DAE has always the fame pofi-
tion, becaufe no other ftraight line
can be drawn through A parallel to
BC: Therefore the ftraight line DAE which has been found is
given ᵇ in pofition.

a 31. 1.

b 4. def.

P R O P.

PROP. XXXII.

IF a ſtraight line be drawn to a given point in a ſtraight line given in poſition, and makes a given angle with it; that ſtraight line is given in poſition.

Let AB be a ſtraight line given in poſition, and C a given point in it, the ſtraight line drawn to C, which makes a given angle with CB, is given in poſition.

Becauſe the angle is given, one equal to it can be found ᵃ; let this be the angle at D, at the given point C, in the given ſtraight line AB, make ᵇ the angle ECB equal to the angle at D: Therefore the ſtraight line EC has always the ſame ſituation, becauſe any other ſtraight line FC, drawn to the point C, makes with CB a greater or leſs angle than the angle ECB, or the angle at D: Therefore the ſtraight line EC, which has been found, is given in poſition.
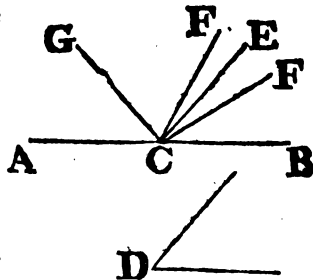
a 1. def.

b 23. 1.

It is to be obſerved, that there are two ſtraight lines EC, GC upon one ſide of AB that make equal angles with it, and which make equal angles with it when produced to the other ſide.

PROP. XXXIII.

30.

IF a ſtraight line be drawn from a given point to a ſtraight line given in poſition, and makes a given angle with it, that ſtraight line is given in poſition.

From the given point A, let the ſtraight line AD be drawn to the ſtraight line BC given in poſition, and make with it a given angle ADC; AD is given in poſition.

Thro' the point A, draw ᵃ the ſtraight line EAF parallel to BC; and becauſe thro' the given point A, the ſtraight line EAF is drawn parallel to BC, which is given in poſition, EAF is therefore given in poſition ᵇ: And becauſe the ſtraight line AD meets the parallels BC, EF, the
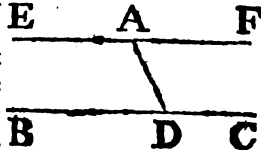
a 31. 1.

b 31. dat.

angle

c 29. 1. angle EAD is equal ᶜ to the angle ADC; and ADC is given,
wherefore alſo the angle EAD is given: Therefore, becauſe the
ſtraight line DA is drawn to the given point A in the ſtraight
line EF given in poſition, and makes with it a given angle EAD,
d 32. dat. AD is given ᵈ in poſition.

3. 1.   P R O P. XXXIV.

See N. IF from a given point to a ſtraight line given in poſition,
a ſtraight line be drawn which is given in magnitude;
the ſame is alſo given in poſition.

Let A be a given point, and BC a ſtraight line given in poſi-
tion, a ſtraight line given in magnitude drawn from the point
A to BC is given in poſition.
Becauſe the ſtraight line is given in magnitude, one equal to
a 1. def. it can be found ᵃ; let this be the ſtraight line D: From the
point A draw AE perpendicular to BC; and
becauſe AE is the ſhorteſt of all the ſtraight
lines which can be drawn from the point A
to BC, the ſtraight line D, to which one
equal is to be drawn from the point A to
BC, cannot be leſs than AE. If therefore D
D be equal to AE, AE is the ſtraight line given in magnitude
drawn from the given point A to BC: And it is evident that
b. 33. dat. AE is given in poſition ᵇ, becauſe it is drawn from the given
point A to BC, which is given in poſition, and makes with BC
the given angle AEC.
But if the ſtraight line D be not equal to AE, it muſt be
greater than it: Produce AE, and make AF equal to D; and
from the centre A, at the diſtance AF, deſcribe the circle GFH,
and join AG, AH: Becauſe the circle GFH is given in poſi-
c 6. def. tion ᶜ, and the ſtraight line BC is alſo given in poſition; there-
d 28. dat. fore their interſection G is given ᵈ;
and the point A is given; where-
e 29. dat. fore AG is given in poſition ᵉ, that
is, the ſtraight line AG given in
magnitude, (for it is equal to D)
and drawn from the given point A
to the ſtraight line BC given in poſition, is alſo given in poſi-
tion: And in like manner AH is given in poſition: Therefore
in this caſe there are two ſtraight lines AG, AH of the ſame
                                                          given
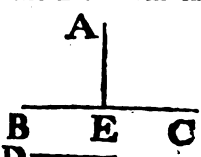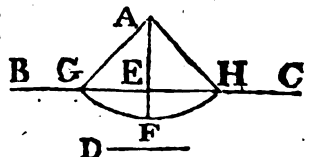
given magnitude which can be drawn from a given point A to
a ſtraight line BC given in poſition.

P R O P.   XXXV.

IF a ſtraight line be drawn between two parallel ſtraight
lines given in poſition, and makes given angles with
them, the ſtraight line is given in magnitude.

Let the ſtraight line EF be drawn between the parallels AB,
CD, which are given in poſition, and make the given angles
BEF, EFD: EF is given in magnitude.
In CD take the given point G, and through G draw ᵃ GH **a** 31. 1.
parallel to EF: And becauſe CD meets the parallels GH, EF,
the angle EFD is equal ᵇ to the angle **b** 29. 1.

And EFD is a given angle;
wherefore the angle HGD is given: And
becauſe HG is drawn to the given point
G, in the ſtraight line CD, given in poſi-
tion, and makes a given angle HGD;
the ſtraight line HG is given in poſi-
tion ᶜ: And AB is given in poſition; therefore the point H is **c** 32. dat.
given ᵈ; and the point G is alſo given, wherefore GH is given **d** 28. dat.
in magnitude ᵉ: And EF is equal to it; therefore EF is given **e** 29. dat.
in magnitude.

P R O P.   XXXVI.

IF a ſtraight line given in magnitude be drawn between See N.
two parallel ſtraight lines given in poſition, it ſhall
make given angles with the parallels.

Let the ſtraight line EF given in magnitude be drawn be-
tween the parallel ſtraight lines AB, CD,
which are given in poſition; the angles
AEF, EFC ſhall be given.
Becauſe EF is given in magnitude, a
ſtraight line equal to it can be found ᵃ; **a** 1. def.
let this be G: In AB take a given point
H, and from it draw ᵇ HK perpendicu- **b** 12. 1.
lar to CD: Therefore the ſtraight line G,

B b 4   that
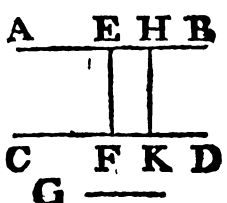
that is, EF cannot be leſs than HK: And if G be equal to HK, EF alſo is equal to it; wherefore EF is at right angles to CD; for if it be not, EF would be greater than HK, which is abſurd: Therefore the angle EFD is a right, and conſequently a given angle.

But if the ſtraight line G be not equal to HK, it muſt be greater than it: Produce HK, and take HL, equal to G; and from the centre H, at the diſtance HL, deſcribe the circle
c 6. def. MLN, and join HM, HN: And becauſe the circle ᶜ MLN, and the ſtraight line CD, are given in poſition, the points M,
d 28. dat. N are ᵈ given; and the point
H is given; wherefore the ſtraight lines HM, HN, are
e 29. dat. given in poſition ᵉ: And CD is given in poſition; therefore the angles HMN, HNM, are
f A. def. given in poſition ᶠ: Of the ſtraight lines HM, HN, let

HN be that which is not parallel to EF, for EF cannot be parallel to both of them; and draw EO parallel to HN: EO thereg 34. 1. fore is equal ᵍ to HN, that is, to G; and EF is equal to G, wherefore EO is equal to EF, and the angle EFO to the angle
h 29. 1. EOF, that is ʰ, to the given angle HNM, and becauſe the angle HNM, which is equal to the angle EFO, or EFD, has been found; therefore the angle EFD, that is, the angle AEF, is given in
k 1. def. magnitude ᵏ; and conſequently the angle EFC.
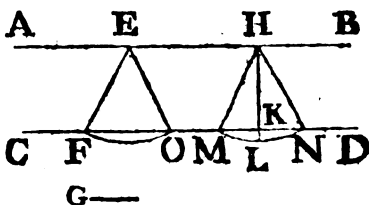
E.   P R O P. XXXVII.

See N. IF a ſtraight line given in magnitude be drawn from a point to a ſtraight line given in poſition, in a given angle; the ſtraight line drawn through that point parallel to the ſtraight line given in poſition, is given in poſition.

Let the ſtraight line AD given in magnitude be drawn from the point A to the ſtraight line BC given in poſition, in the given angle ADC; the ſtraight line EAF drawn through A parallel to BC is given in poſition.

In BC take a given point G, and draw GH parallel to AD: And becauſe HG is drawn to a given point G in the ſtraight line BC gi

ven in pofition, in a given angle HGC, for it is equal [a] to the
given angle ADC; HG is given in pofition [b]; but it is given
alfo in magnitude, becaufe it is equal to [c] AD which is given in
magnitude; therefore becaufe G one of the extremities of the
ftraight line GH given in pofition and magnitude is given, the
other extremity H is given [d]; and the ftraight line EAF, which
is drawn through the given point H parallel to BC given in po-
fition, is therefore given [e] in pofition.

a 29. 1.
b 32. dat.
c 34. 1.

d 30. dat.

e 31. dat.

P R O P. XXXVIII.

34.

IF a ftraight line be drawn from a given point to two
parallel ftraight lines given in pofition, the ratio of the
fegments between the given point and the parallels fhall
be given.

Let the ftraight line EFG be drawn from the given point E
to the parallels AB, CD, the ratio of EF to EG is given.

From the point E draw EHK perpendicular to CD; and
becaufe from a given point E the ftraight line EK is drawn to
CD which is given in pofition, in a given angle EKC; EK is
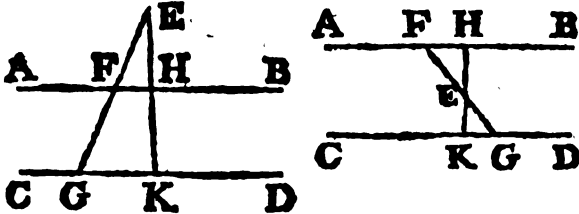
given in pofition [a]; and AB, CD are given in pofition; there-
fore [b] the points H, K are given: And the point E is given;
wherefore [c] EH, EK are given in magnitude, and the ratio [d] of
them is therefore given. But as EH to EK, fo is EF to EG, be-
caufe AB, CD are parallels; therefore the ratio of EF to EG is
given.

a 33. dat.
b 28. dat.
c 29. dat.
d 1. dat.

P R O P. XXXIX.

35. 36.

IF the ratio of the fegments of a ftraight line between See N.
a given point in it and two parallel ftraight lines, be
given, if one of the parallels be given in pofition, the
other is alfo given in pofition.

From

From the given point A, let the ftraight line AED be drawn
to the two parallel ftraight lines FG, BC; and let the ratio of
the fegments AE, AD be given ; if one of the parallels BC be
given in pofition, the other FG is alfo given in pofition.

From the point A, draw AH perpendicular to BC, and let
it meet FG in K ; and becaufe AH is drawn from the given
point A to the ftraight line BC given in pofition, and makes a
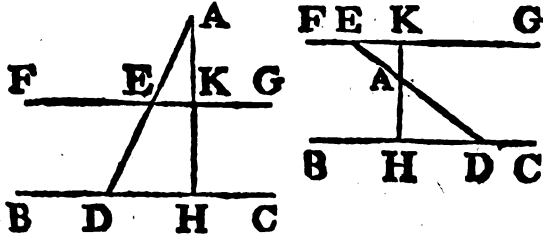
a 33. dat.
given angle AHD ; AH is given ᵃ in
pofition ; and BC is likewife given in
pofition, therefore the point H is gi-

b 28. dat.
ven ᵇ : The point A is alfo given ;

c 29. dat.
wherefore AH is given in magnitude ᶜ,
and, becaufe FG, BC are parallels,
as AE to AD, fo is AK to AH ; and
the ratio of AE to AD is given,
wherefore the ratio of AK to AH is given ; but AH is given

d 2. dat.
in magnitude, therefore ᵈ AK is given in magnitude ; and it is

e 30. dat.
alfo given in pofition, and the point A is given ; wherefore ᵉ
the point K is given. – And becaufe the ftraight line FG is drawn
through the given point K parallel to BC which is given in po-

f 31. dat.
fition, therefore ᶠ FG is given in pofition.

37. 38.

P R O P. XL.

See N.
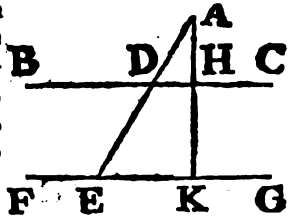
IF the ratio of the fegments of a ftraight line into
which it is cut by three parallel ftraight lines, be gi-
ven ; if two of the parallels are given in pofition, the
third alfo is given in pofition.

Let AB, CD, HK be three parallel ftraight lines, of which
AB, CD are given in pofition ; and let the ratio of the feg-
ments

ments GE, GF into which the ſtraight line GEF is cut by the three parallels, be given ; the third parallel HK is given in poſition.

In AB take a given point L, and draw LM perpendicular to CD, meeting HK in N ; becauſe LM is drawn from the given point L to CD which is given in poſition and makes a given angle LMD ; LM is given in poſition [a] ; and CD is given in poſition, wherefore the point M is given [b] ; and the point L is given, LM is therefore given in magnitude [c] ; and becauſe the ratio of GE to GF is given, and as GE to GF, ſo is NL to

a 33. dat.
b 28. dat.
c 29. dat.
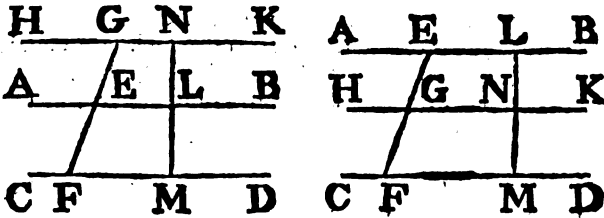
NM ; the ratio of NL to NM is given ; and therefore [d] the ratio of ML to LN is given ; but LM is given in magnitude, wherefore [e] LN is given in magnitude ; and it is alſo given in poſition, and the point L is given ; wherefore [f] the point N is given ; and becauſe the ſtraight line HK is drawn through the given point N parallel to CD which is given in poſition, therefore HK is given in poſition [g].

Cor.
d { 6. or
{ 7. dat.
e 2. dat.
f 30. dat.

g 31. dat.

P R O P. XLI. F.

IF a ſtraight line meets three parallel ſtraight lines See N. which are given in poſition, the ſegments into which they cut it have a given ratio.

Let the parallel ſtraight lines AB, CD, EF given in poſition, be cut by the ſtraight line GHK ; the ratio of GH to HK is given.

In AB take a given point L, and draw LM perpendicular to CD, meeting EF in N ; therefore [a] LM is given in poſition ; and CD, EF are given in poſition, wherefore the points M, N are given : And the point L is given ; therefore [b] the ſtraight lines LM, MN are given in magnitude ; and the ratio

a 33. dat.

b 29. dat.

of
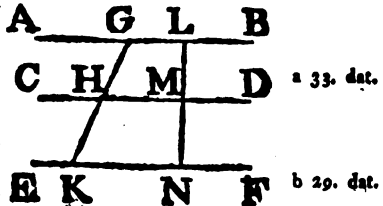
of LM to MN is therefore given ^e : But as LM to MN, so is GH to HK; wherefore the ratio of GH to HK is given.

PROP. XLII.

39.

ice N. IF each of the sides of a triangle be given in magnitude, the triangle is given in species.

Let each of the sides of the triangle ABC be given in magnitude, the triangle ABC is given in species.

a 22. 1.   Make a triangle ^a DEF the sides of which are equal, each to each, to the given straight lines AB, BC, CA, which can be done, because any two of them must be greater than the third; and let DE be e-
qual to AB, EF to BC, and FD to CA; and be-
cause the two sides ED, DF are equal to the two BA, AC, each to each, and the base EF equal to
b 8. 1. the base BC; the angle
EDF is equal ^b to the angle BAC; therefore, because the angle EDF, which is equal to the angle BAC, has been found, the
c 1. def. angle BAC is given ^c, in like manner the angles at B, C are given. And because the sides AB, BC, CA are given, their
d 1. dat. ratios to one another are given ^d, therefore the triangle ABC
e 3. def. is given ^e in species.
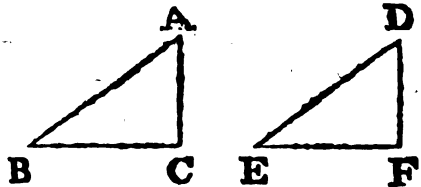
PROP. XLIII.

40.

IF each of the angles of a triangle be given in magni-
tude, the triangle is given in species.

Let each of the angles of the triangle ABC be given in mag-
nitude, the triangle ABC is given in species.

Take a straight line DE given in
position and magnitude, and at the
a 23. 1. points D, E make ^a the angle EDF
equal to the angle BAC, and the
angle DEF equal to ABC; there-
fore the other angles EFD, BCA
are equal, and each of the angles at the points A, B, C, is gi-
ven,
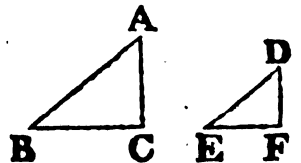
ven, wherefore each of thofe at the points D, E, F is given:
And becaufe the ftraight line FD is drawn to the given point
D in DE which is given in pofition, making the given angle
EDF; therefore DF is given in pofition [b].   In like manner EF b 32. dat.
alfo is given in pofition; wherefore the point F is given: And
the points D, E are given; therefore each of the ftraight lines
DE, EF, FD is given [c] in magnitude; wherefore the triangle c 29. dat.
DEF is given in fpecies [d]; and it is fimilar [e] to the triangle d 42. dat.
ABC; which therefore is given in fpecies. e { 4. 6. / 1. def. / 6.

P R O P. XLIV.
41.

IF one of the angles of a triangle be given, and if the
fides about it have a given ratio to one another; the
triangle is given in fpecies.

Let the triangle ABC have one of its angles BAC given, and
let the fides BA, AC about it have a given ratio to one another;
the triangle ABC is given in fpecies.
Take a ftraight line DE given in pofition and magnitude,
and at the point D in the given ftraight line DE, make the
angle EDF equal to the given angle BAC; wherefore the
angle EDF is given; and becaufe the ftraight line FD is drawn
to the given point D in ED which is given in pofition, making
the given angle EDF; therefore FD
is given in pofition [a].   And becaufe a 32. dat.
the ratio of BA to AC is given,
make the ratio of ED to DF the
fame with it, and join EF; and be-
caufe the ratio of ED to DF is gi-
ven, and ED is given, therefore [b] DF is given in magnitude; b 2. dat.
and it is given alfo in pofition, and the point D is given, where-
fore the point F is given [c]; and the points D, E are given, c 30. dat.
wherefore DE, EF, FD are given [d] in magnitude; and the d 29. dat.
triangle DEF is therefore given [e] in fpecies; and becaufe the e 42. dat.
triangles ABC, DEF have one angle BAC equal to one angle
EDF, and the fides about thefe angles proportionals; the tri-
angles are [f] fimilar; but the triangle DEF is given in fpecies, f 6. 6.
and therefore alfo the triangle ABC.
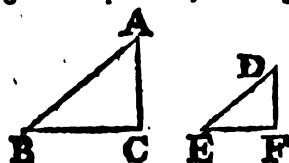
P R O P.

P R O P. XLV.

See N.

IF the fides of a triangle have to one another given ra-
tios; the triangle is given in fpecies.

Let the fides of the triangle ABC have given ratios to one
another, the triangle ABC is given in fpecies.

Take a ftraight line D given in magnitude; and becaufe the
ratio of AB to BC is given, make the ratio of D to E the
fame with it; and D is given, therefore [a] E is given. And be-
caufe the ratio of BC to CA is given, to this make the ratio
of E to F the fame; and E is given, and therefore [a] F; and
becaufe as AB to BC, fo is D to E; by compofition AB and
BC together are to BC, as D
and E to E; but as BC to CA,
fo is E to F; therefore, ex ae-
quali [b], as AB and BC are to
CA, fo are D and E to F, and
AB and BC are greater [c] than
CA; therefore D and E are
greater [d] than F. In the fame
manner any two of the three D,
E, F are greater than the third.
Make [e] the triangle GHK whofe
fides are equal to D, E, F, fo that GH be equal to D, HK to
E, and KG to F; and becaufe D, E, F, are, each of them,
given, therefore GH, HK, KG are each of them given in mag-
nitude; therefore the triangle GHK is given [f] in fpecies: But
as AB to BC, fo is (D to E, that is) GH to HK; and as BC to
CA, fo is (E to F, that is) HK to KG; therefore, ex aequali,
as AB to AC, fo is GH to GK. Wherefore [g] the triangle ABC
is equiangular and fimilar to the triangle GHK; and the tri-
angle GHK is given in fpecies; therefore alfo the triangle ABC
is given in fpecies.
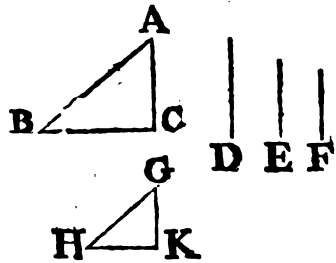
Cor. If a triangle is required to be made, the fides of which
fhall have the fame ratios which three given ftraight lines D,
E, F have to one another; it is neceffary that every two of
them be greater than the third.

a 2. dat.

b 22. 5.

c 20. 1.

d A. 5.

e 22. 1.

f 42. dat.

g 5. 6.

P R O P.

P R O P. XLVI.

IF the fides of a right angled triangle about one of the
acute angles have a given ratio to one another; the
triangle is given in fpecies.

Let the fides AB, BC about the acute angle ABC of the tri-
angle ABC, which has a right angle at A, have a given ratio to
one another; the triangle ABC is given in fpecies.
Take a ftraight line DE given in pofition and magnitude;
and becaufe the ratio of AB to BC is given, make as AB to
BC, fo DE to EF; and becaufe DE has a given ratio to EF,
and DE is given, therefore[a] EF is given; and becaufe as AB [a] 2. dat.
to BC, fo is DE to EF; and AB is lefs[b] than BC, therefore [b] 19. 1.
DE is lefs[c] than EF. From the point D draw DG at right angles [c] A. 5.
to DE, and from the centre E
at the diftance EF, defcribe a
circle which fhall meet DG in
two points; let G be either of
them, and join EG; there-
fore the circumference of the
circle is given[d] in pofition;

[d] 6. def.

and the ftraight line DG is given[e] in pofition, becaufe it is [e] 32. dat.
drawn to the given point D in DE given in pofition, in a given
angle; therefore[f] the point G is given; and the points D, E [f] 28. dat.
are given, wherefore DE, EG, GD are given[g] in magnitude, [g] 29. dat.
and the triangle DEG in fpecies[h]. And becaufe the triangles [h] 42. dat.
ABC, DEG have the angle BAC equal to the angle EDG,
and the fides about the angles ABC, DEG proportionals, and
each of the other angles BCA, EGD lefs than a right angle;
the triangle ABC is equiangular[i] and fimilar to the triangle [i] 7. 6.
DEG: But DEG is given in fpecies; therefore the triangle
ABC is given in fpecies: And in the fame manner, the triangle
made by drawing a ftraight line from E to the other point in
which the circle meets DG is given in fpecies.

P R O P.

44.
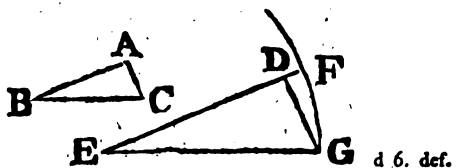
See N.

IF a triangle has one of its angles which is not a right angle given, and if the sides about another angle have a given ratio to one another; the triangle is given in species.

Let the triangle ABC have one of its angles ABC a given, but not a right angle, and let the sides BA, AC about another angle BAC have a given ratio to one another; the triangle ABC is given in species.

Firſt, Let the given ratio be the ratio of e-quality, that is, let the sides BA, AC, and conſequently the angles ABC, ACB, be equal ; and becauſe the angle ABC is given, the

a 32. 1.

angle ACB, and alſo the remaining [a] angle BAC is given ; therefore the triangle ABC is

b 43. dat.

given [b] in ſpecies : And it is evident that in this caſe the given angle ABC muſt be acute.
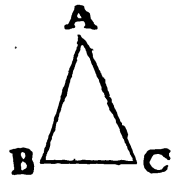
Next, Let the given ratio be the ratio of a leſs to a greater, that is, let the ſide AB adjacent to the given angle be leſs than the ſide AC : Take a ſtraight line DE given in poſition and magnitude, and make the angle DEF equal to the given

c 32. dat.

angle ABC; therefore EF is given [c] in poſition; and becauſe the ratio of BA to AC is given, as BA to AC, ſo make ED to DG; and becauſe the ratio of ED to DG is given, and ED is given, the ſtraight line DG is gi-

d 2. dat.

ven [d], and BA is leſs than AC,

e A. 5.

therefore ED is leſs [e] than DG. From the centre D, at the diſtance DG deſcribe the circle GF meet-ing EF in F, and join DF ; and

f 6. def.

becauſe the circle is given [f] in po-ſition, as alſo the ſtraight line EF,

g 28. dat.

the point F is given [g]; and the points D, E are given, wherefore the ſtraight lines DE, EF, FD
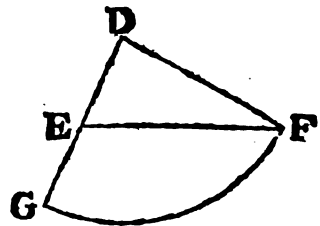
h 29. dat.

are given [h] in magnitude, and the triangle DEF in ſpe-

i 42. dat.

cies [i], and becauſe BA is leſs than AC, the angle ACB is

k 18. 1.

leſs [k] than the angle ABC, and therefore ACB is leſs [l] than

l 1. 7. 1.

2

a right angle. In the fame manner, becaufe ED is lefs than DG or DF, the angle DFE is lefs than a right angle : And becaufe the triangles ABC, DEF have the angle ABC equal to the angle DEF, and the fides about the angles BAC, EDF proportionals, and each of the other angles ACB, DFE lefs than a right angle; the triangles ABC, DEF are [m] fimilar, and DEF is given in fpecies, wherefore the triangle ABC is alfo given in fpecies.

[m] 7. 6.

Thirdly, Let the given ratio be the ratio of a greater to a lefs, that is, let the fide AB adjacent to the given angle be greater than AC; and, as in the laft cafe, take a ftraight line DE given in pofition and magnitude, and make the angle DEF equal to the given angle ABC; therefore EF is given [c] in pofition : Alfo draw DG perpendicular to EF; therefore if the ratio of BA to AC be the fame with the ratio of ED to the perpendicular DG, the triangles ABC, DEG are fimilar [m], becaufe the angles ABC, DEG are equal, and DGE is a right angle : Therefore the angle ACB is a right angle, and the triangle ABC is given [b] in fpecies.
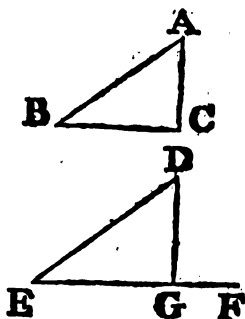
[c] 32. dat.

[b] 43. dat.

But if, in this laft cafe, the given ratio of BA to AC be not the fame with the ratio of ED to DG, that is, with the ratio of BA to the perpendicular AM drawn from A to BC; the ratio of BA to AC muft be lefs [o] than the ratio of BA to AM, becaufe AC is greater than AM. Make as BA to AC fo ED to DH; therefore the ratio of ED to DH is lefs than the ratio of (BA to AM, that is, than the ratio of) ED to DG; and confequently, DH is greater [p] than DG; and becaufe BA is greater than AC, ED is greater [e] than DH. From the centre D, at the diftance DH, defcribe the circle KHF which neceffarily meets the ftraight line EF in two points, becaufe DH is greater than DG, and lefs than DE. Let the circle meet EF in the points F, K which are given, as was fhown in the preceding cafe; and, DF, DK being joined, the triangles DEF, DEK are given in fpecies, as was there fhewn.

[o] 8. 5.

[p] 10. 5.

[e] A. 5.

C c
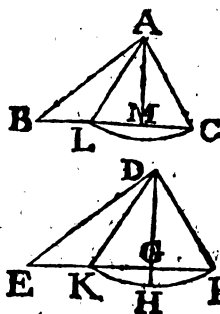
shewn. From the centre A, at the distance AC, describe a circle meeting BC again in L: And if the angle ACB be less than a right angle, ALB must be greater than a right angle: And on the contrary. In the same manner, if the angle DFE be less than a right angle, DKE must be greater than one; and on the contrary. Let each of the angles ACB, DFE be either less or greater than a right angle; and because in the triangles ABC, DEF the angles ABC, DEF are equal, and the sides BA, AC, and ED, DF about two of the other angles proportionals, the triangle ABC is similar [m] to the triangle DEF. In the same manner, the triangle ABL is similar to DEK. And the triangles, DEF, DEK are given in species; therefore also the triangles ABC, ABL are given in species. And from this it is evident, that, in this third case, there are always two triangles of a different species, to which the things mentioned as given in the proposition can agree.

m. 7. 6.

PROP. XLVIII.

45.
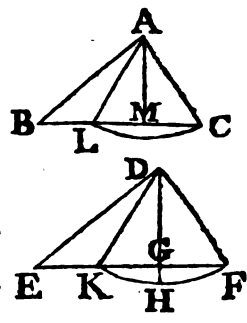
IF a triangle has one angle given, and if both the sides together about that angle have a given ratio to the remaining side; the triangle is given in species.

Let the triangle ABC have the angle BAC given, and let the sides BA, AC together about that angle have a given ratio to BC; the triangle ABC is given in species.

Bisect [a] the angle BAC by the straight line AD; therefore the angle BAD is given. And because as BA to AC, so is [b] BD to DC, by permutation, as AB to BD, so is AC to CD; and as BA and AC together to BC, so is [c] AB to BD. But the ratio of BA and AC together to BC is given, wherefore the ratio of AB to BD is given, and the angle BAD is given; therefore [d] the triangle ABD is given in species, and the angle ABD is therefore given; the angle BAC is also given, wherefore the triangle ABC is given in species [e].
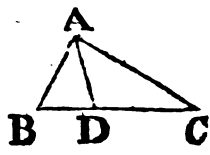
A triangle which shall have the things that are mentioned in the proposition to be given, can be found in the following manner.

a 9. 1.
b 3. 6.
c 12. 5.
d 47. dat.
e 43. dat.

manner. Let EFG be the given angle, and let the ratio of H to K be the given ratio which the two fides about the angle EFG muft have to the third fide of the triangle; therefore becaufe two fides of a triangle are greater than the third fide, the ratio of H to K muft be the ratio of a greater to a lefs. Bifect[a] the angle EFG by the ftraight line FL, and by the 47th propofition find a triangle of which EFL is one of the angles, and in which the ratio of the fides about the angle oppofite to FL is the fame with the ratio of H to K: To do which, take FE given in pofition and magnitude, and draw EL perpendicular to FL: Then if the ratio of H to K be the fame with the ratio of FE to EL, produce EL, and let it meet FG in P; the triangle FEP is that which was to be found: For it has the given angle EFG; and becaufe this angle is bifected by FL, the fides EF, FP together are to EP, as [b] FE to EL, that is, as H to K.
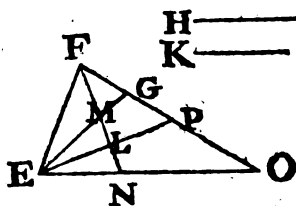
But if the ratio of H to K be not the fame with the ratio of FE to EL, it muft be lefs than

a 9. 1.

b 3. 6.

it, as was fhown in prop. 47. and in this cafe there are two triangles, each of which has the given angle EFL, and the ratio of the fides about the angle oppofite to FL the fame with the ratio of H to K. By prop. 47. find thefe triangles EFM, EFN each of which has the angle EFL for one of its angles, and the ratio of the fide FE to EM or EN the fame with the ratio of H to K; and let the angle EMF be greater, and ENF lefs than a right angle. And becaufe H is greater than K, EF is greater than EN, and therefore the angle EFN, that is, the angle NFG, is lefs[f] than the angle ENF. To each of thefe add the angles[f] 18. 1. NEF, EFN; therefore the angles NEF, EFG are lefs than the angles NEF, EFN, FNE, that is, than two right angles; therefore the ftraight lines EN, FG muft meet together when produced; let them meet in O, and produce EM to G. Each of the triangles, EFG, EFO has the things mentioned to be given in the propofition: For each of them has the given angle EFG; and becaufe this angle is bifected by the ftraight line FMN, the fides EF, FG together have to EG the third fide the ratio of FE to EM, that is, of H to K. In like manner, the fides EF, FO together have to EO the ratio which H has to K.

PROP.

P R O P. XLIX.

IF a triangle has one angle given, and if the fides a-
bout another angle, both together, have a given ra-
tio to the third fide; the triangle is given in fpecies.

Let the triangle ABC have one angle ABC given, and let the
two fides BA, AC about another angle BAC have a given ratio
to BC; the triangle ABC is given in fpecies.

Suppofe the angle BAC to be bifected by the ftraight line
AD; BA and AC together are to BC, as AB to BD, as was
fhown in the preceding propofition. But the ratio of BA and
AC together to BC is given, therefore alfo the ratio of AB to

a 44. dat. BD is given. And the angle ABD is given, wherefore ᵃ the
triangle ABD is given in fpecies; and confequently the angle
BAD, and its double the angle BAC
are given; and the angle ABC is gi-
ven. Therefore the triangle ABC is

b 43. dat. given in fpecies ᵇ.

A triangle which fhall have the
things mentioned in the propofition to
be given, may be thus found. Let
EFG be the given angle, and the ra-
tio of H to K the given ratio; and
by prop. 44. find the triangle EFL,
which has the angle EFG for one of
its angles, and the ratio of the fides
EF, FL about this angle the fame with
the ratio of H to K; and make the angle LEM equal to the
angle FEL. And becaufe the ratio of H to K is the ratio which
two fides of a triangle have to the third, H muft be greater than
K; and becaufe EF is to FL, as H to K, therefore EF is greater
than FL, and the angle FEL, that is, LEM, is therefore lefs than
the angle ELF. Wherefore the angles LFE, FEM are lefs than
two right angles, as was fhown in the foregoing propofition, and
the ftraight lines FL, EM muft meet if produced; let them
meet in G, EFG is the triangle which was to be found; for
EFG is one of its angles, and becaufe the angle FEG is bifected
by EL, the two fides FE, EG together have to the third fide
FG the ratio of EF to FL, that is, the given ratio of H to K.
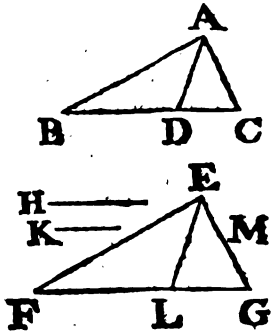
P R O P.

PROP. L.

76.

IF from the vertex of a triangle given in fpecies, a ftraight line be drawn to the bafe in a given angle; it fhall have a given ratio to the bafe.

From the vertex A of the triangle ABC which is given in fpecies, let AD be drawn to the bafe BC in a given angle ADB; the ratio of AD to BC is given.

Becaufe the triangle ABC is given in fpecies, the angle ABD is given, and the angle ADB is given; therefore the triangle ABD is given [a] in fpecies; wherefore the ratio of AD to AB is given. And the ratio of AB to BC is given; and therefore [b] the ratio of AD to BC is given.
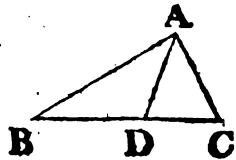
a 43. dat.

b 9. dat.

PROP. LI.

47.

RECTILINEAL figures given in fpecies, are divided into triangles which are given in fpecies.

Let the rectilineal figure ABCDE be given in fpecies; ABCDE may be divided into triangles given in fpecies.

Join BE, BD; and becaufe ABCDE is given in fpecies, the angle BAE is given [a], and the ratio of BA to BE is given [a]; wherefore the triangle BAE is given in fpecies [b], and the angle AEB is therefore given [a]. But the whole angle AED is given, and therefore the remaining angle BED is given, and the ratio of AE to EB is given, as alfo the ratio of AE to ED; therefore the ratio of BE to ED is given [c]. And the angle BED is given, wherefore the triangle BED is given [b] in fpecies. In the fame manner, the triangle BDC is given in fpecies: Therefore rectilineal figures which are given in fpecies are divided into triangles given in fpecies.
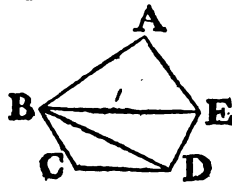
a 3. def.

b 44. dat.

c 9. dat.

PROP.

48.

P R O P. LII.

IF two triangles given in fpecies be defcribed upon the fame ftraight line; they fhall have a given ratio to one another.

Let the triangles ABC, ABD given in fpecies be defcribed upon the fame ftraight line AB; the ratio of the triangle ABC to the triangle ABD is given.

Through the point C, draw CE parallel to AB, and let it meet DA produced in E, and join BE. Becaufe the triangle ABC is given in fpecies, the angle BAC, that is, the angle ACE, is given; and becaufe the triangle ABD is given in fpe-cies, the angle DAB, that is, the angle AEC, is given. Therefore the triangle ACE is given in fpecies; wherefore the ratio of EA to AC is given [a], and the ra-tio of CA to AB is given, as alfo the ra-tio of BA to AD; therefore the ratio of [b] EA to AD is given, and the triangle ACB is equal [c] to the triangle AEB, and as the triangle AEB, or ACB, is to the triangle ADB, fo is [d] the ftraight line EA to AD. But the ratio of EA to AD is given, therefore the ratio of the triangle ACB to the triangle ADB is given.
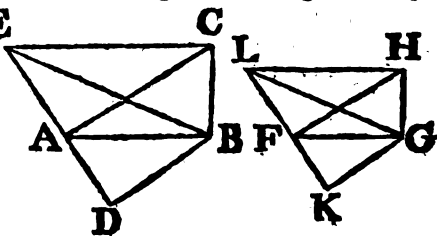
a 3. def.
b 9. dat.
c 37. 1.
d 1. 6.

P R O B L E M.

To find the ratio of two triangles ABC, ABD given in fpe-cies, and which are defcribed upon the fame ftraight line AB.

Take a ftraight line FG given in pofition and magnitude, and becaufe the angles of the triangles ABC, ABD are given, at the points F, G of the ftraight line FG, make the angles GFH, GFK [e] equal to the angles BAC, BAD; and the angles FGH, FGK equal to the angles ABC, ABD, each to each. Therefore the triangles ABC, ABD are equiangular to the tri-angles FGH, FGK, each to each. Through the point H draw HL parallel to FG meeting KF produced in L. And becaufe the angles BAC, BAD are equal to the angles GFH, GFK, each to each; therefore the angles ACE, AEC are equal to FHL, FLH, each to each, and the triangle AEC equiangular to the triangle FLH. Therefore as EA to AC, fo is LF to FH; and

e 23. 1.

as

as CA to AB, fo HF to FG ; and as BA to AD, fo is GF
to FK ; wherefore, ex aequali, as EA to AD, fo is LF to
FK. But, as was fhown, the triangle ABC is to the triangle
ABD, as the ftraight line EA to AD, that is, as LF to FK.
The ratio therefore of LF to FK has been found, which is the
fame with the ratio of the triangle ABC to the triangle ABD.

P R O P. LIII.

49.

IF two rectilineal figures given in fpecies be defcribed See N.
upon the fame ftraight line ; they fhall have a given
ratio to one another.

Let any two rectilineal figures ABCDE, ABFG which are
given in fpecies, be defcribed upon the fame ftraight line AB ;
the ratio of them to one another is given.
Join AC, AD, AF ; each of the triangles AED, ADC,
ACB, AGF, ABF is given [a] in fpecies. And becaufe the tri-   [a] 51. dat.

angles ADE, ADC given in fpecies
are defcribed upon the fame ftraight
line AD, the ratio of EAD to DAC
is given [b] ; and, by compofition,
the ratio of EACD to DAC is
given [c]. And the ratio of DAC to
CAB is given [b], becaufe they are
defcribed upon the fame ftraight
line AC ; therefore the ratio of
EACD to ACB is given [d] ; and, by
compofition, the ratio of ABCDE
to ABC is given. In the fame manner, the ratio of ABFG to
ABF is given. But the ratio of the triangle ABC to the triangle
ABF is given ; wherefore [b], becaufe the ratio of ABCDE to
ABC is given, as alfo the ratio of ABC to ABF, and the ratio
of ABF to ABFG ; the ratio of the rectilineal ABCDE to the
rectilineal ABFG is given [d].
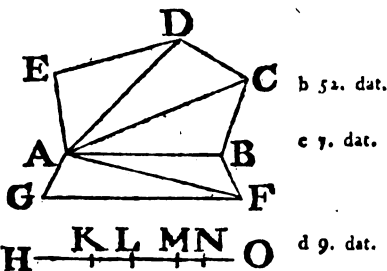
[b] 52. dat.

[c] 7. dat.

[d] 9. dat.

P R O B L E M.

To find the ratio of two rectilineal figures given in fpecies,
and defcribed upon the fame ftraight line.
Let ABCDE, ABFG be two rectilineal figures given in
fpecies, and defcribed upon the fame ftraight line AB, and
join AC, AD, AF. Take a ftraight line HK given in pofition
and magnitude, and by the 52d dat. find the ratio of the tri-
angle ADE to the triangle ADC, and make the ratio of HK

to KL the fame with it. Find alfo the ratio of the triangle
ACD to the triangle ACB. And make the ratio of KL to
LM the fame. Alfo, find the ratio of the triangle ABC to the
triangle ABF, and make the ratio of LM to MN the fame.
And laftly, find the ratio of the triangle AFB to the triangle
AFG, and make the ratio of MN
to NO the fame. Then the ratio of
ABCDE to ABFG is the fame with
the ratio of HM to MO.
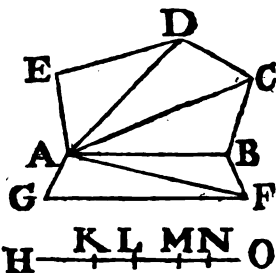
Becaufe the triangle EAD is to
the triangle DAC, as the ftraight
line HK to KL; and as the triangle
DAC to CAB, fo is the ftraight
line KL to LM; therefore, by ufing
compofition as often as the number
of triangles requires, the rectilineal
ABCDE is to the triangle ABC, as the ftraight line HM to ML.
In like manner, becaufe the triangle GAF is to FAB, as ON to
NM, by compofition, the rectilineal ABFG is to the triangle ABF,
as MO to MN; and, by inverfion, as ABF to ABFG, fo is NM
to MO. And the triangle ABC is to ABF, as LM to MN.
Wherefore, becaufe as ABCDE to ABC, fo is HM to ML; and
as ABC to ABF, fo is LM to MN; and as ABF to ABFG, fo
is MN to MO; ex aequali, as the rectilineal ABCDE to ABFG,
fo is the ftraight line HM to MO.

PROP. LIV.

50.

IF two ftraight lines have a given ratio to one another;
the fimilar rectilineal figures defcribed upon them fimi-
larly, fhall have a given ratio to one another.

Let the ftraight lines AB, CD, have a given ratio to one an-
other, and let the fimilar and fimilarly placed rectilineal figures
E, F be defcribed upon them; the ratio of E to F is given.

To AB, CD, let G be a third
proportional; therefore as AB to
CD, fo is CD to G. And the ratio
of AB to CD is given, wherefore
the ratio of CD to G is given; and
confequently the ratio of AB to G is
alfo given ᵃ. But as AB to G, fo is
the figure E to the figure ᵇ F. Therefore the ratio of E to F is
given.
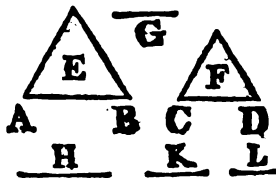
a 9. dat.
b 2. cor. 20.
6.

PROBLEM.

PROBLEM.

To find the ratio of two ſimilar rectilineal figures, E, F, ſimilarly deſcribed upon ſtraight lines AB, CD which have a given ratio to one another: Let G be a third proportional to AB, CD.

Take a ſtraight line H given in magnitude; and becauſe the ratio of AB to CD is given, make the ratio of H to K the ſame with it; and becauſe H is given, K is given. As H is to K, ſo make K to L; then the ratio of E to F is the ſame with the ratio of H to L; for AB is to CD, as H to K, wherefore CD is to G, as K to L; and, ex aequali, as AB to G, ſo is H to L: But the figure E is to [b] the figure F, as AB to G, that is, as H to L.

b 2 cor. 20. 6.

PROP. LV.

51.

IF two ſtraight lines have a given ratio to one another; the rectilineal figures given in ſpecies deſcribed upon them, ſhall have to one another a given ratio.

Let AB, CD be two ſtraight lines which have a given ratio to one another; the rectilineal figures E, F given in ſpecies and deſcribed upon them, have a given ratio to one another.

Upon the ſtraight line AB, deſcribe the figure AG ſimilar and ſimilarly placed to the figure F; and becauſe F is given in ſpecies, AG is alſo given in ſpecies: Therefore, ſince the figures E, AG which are given in ſpecies, are deſcribed upon the ſame ſtraight line, AB, the ratio of E to AG is given [a], and becauſe the ratio of AB to CD is given,

a 53. dat.

and upon them are deſcribed the ſimilar and ſimilarly placed rectilineal figures AG, F, the ratio of AG to F is given [b]; and the ratio of AG to E is given; therefore the ratio of E to F is given [c].
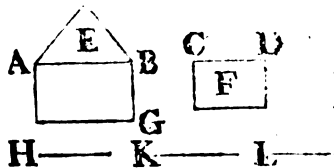
b 54. dat.

c 9. dat.

PROBLEM.

To find the ratio of two rectilineal figures E, F given in ſpecies, and deſcribed upon the ſtraight lines AB, CD which have a given ratio to one another.

Take a ſtraight line H given in magnitude; and becauſe the rectilineal figures E, AG given in ſpecies are deſcribed upon the ſame ſtraight line AB, find their ratio by the 53d dat. and make the ratio of H to K the ſame; K is therefore given: And becauſe the ſimilar rectilineal figures AG, F are deſcribed

upon

upon the ſtraight lines AB, CD, which have a given ratio, find their ratio by the 54th dat. and make the ratio of K to L the ſame: The figure E has to F the ſame ratio which H has to L: For, by the conſtruction, as E is to AG, ſo is H to K; and as AG to F, ſo is K to L; therefore, ex aequali, as E to F; ſo is H to L.

52.

P R O P. LVI.

IF a rectilineal figure given in ſpecies be deſcribed upon a ſtraight line given in magnitude; the figure is given in magnitude.

Let the rectilineal figure ABCDE given in ſpecies be deſcribed upon the ſtraight line AB given in magnitude; the figure ABCDE is given in magnitude.

Upon AB let the ſquare AF be deſcribed; therefore AF is given in ſpecies and magnitude, and becauſe the rectilineal figures ABCDE, AF given in ſpecies are deſcribed upon the ſame ſtraight line AB, the ratio of ABCDE to AF is given [a]:

a 53. dat.

But the ſquare AF is given in magnitude,

b 2. dat.

therefore [b] alſo the figure ABCDE is given in magnitude.
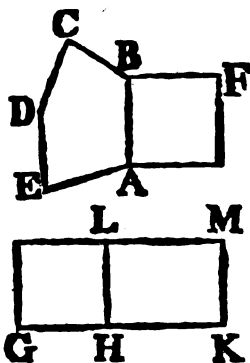
P R O B.

To find the magnitude of a rectilineal figure given in ſpecies deſcribed upon a ſtraight line given in magnitude.

Take the ſtraight line GH equal to the given ſtraight line AB, and by the 53d dat. find the ratio which the ſquare AF upon AB has to the figure ABCDE; and make the ratio of GH to HK the ſame; and upon GH deſcribe the ſquare GL, and complete the parallelogram LHKM; the figure ABCDE is equal to LHKM: Becauſe AF is to ABCDE, as the ſtraight line GH to HK, that is, as the figure GL to HM; and AF is

c 14. 5.

equal to GL; therefore ABCDE is equal to HM [c].

53.

P R O P. LVII.

IF two rectilineal figures are given in ſpecies, and if a ſide of one of them has a given ratio to a ſide of the other; the ratios of the remaining ſides to the remaining ſides ſhall be given.

Let

Let AC, DF be two rectilineal figures given in species, and let the ratio of the side AB to the side DE be given, the ratios of the remaining sides to the remaining sides are also given.

Because the ratio of AB to DE is given, as also [a] the ratios of AB to BC, and of DE to EF, the ratio of BC to EF is given [b]. In the same manner, the ratios of the other sides to the other sides are given.

a 3. def.

b 10. dat.

The ratio which BC has to EF may be found thus : Take a straight line G given in magnitude, and because the ratio of BC to BA is given, make the ratio of G to H the same ; and because the ratio of AB to DE is given, make the ratio of H to K the same; and make the ratio of K to L the same with the given ratio of DE to EF. Since therefore as BC to BA, so is G to H ; and as BA to DE, so is H to K; and as DE to EF, so is K to L ; ex aequali, BC is to EF, as G to L ; therefore the ratio of G to L has been found, which is the same with the ratio of BC to EF.
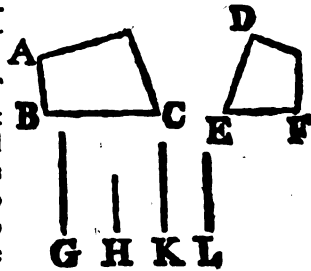
P R O P.   LVIII.

See N.

IF two similar rectilineal figures have a given ratio to one another, their homologous sides have also a given ratio to one another.

Let the two similar rectilineal figures A, B have a given ratio to one another, their homologous sides have also a given ratio.

Let the side CD be homologous to EF, and to CD, EF let the straight line G be a third proportional. As therefore [a] CD to G, so is the figure A to B ; and the ratio of A to B is given, therefore the ratio of CD to G is given ; and CD, EF, G are proportionals ; wherefore [b] the ratio of CD to EF is given.
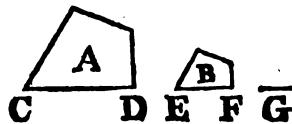
a 2. Cor. 20. 6.

b 13. dat.

The ratio of CD to EF may be found thus : Take a straight line H given in magnitude ; and because the ratio of the figure A to B is given, make the ratio of H to K the same with it : And, as the 13th dat. directs to be done, find a mean proportional L between

G.

between

between H and K; the ratio of CD to EF is the same with that of H to L. Let G be a third proportional to CD, EF; therefore as CD to G, so is (A to B, and so is) H to K; and as CD to EF, so is H to L, as is shewn in the 13th dat.

54.

P R O P. LIX.

See N.

IF two rectilineal figures given in species have a given ratio to one another, their sides shall likewise have given ratios to one another.

Let the two rectilineal figures A, B given in species, have a given ratio to one another, their sides shall also have given ratios to one another.

If the figure A be similar to B, their homologous sides shall have a given ratio to one another, by the preceding proposition; and because the figures are given in species, the sides of each of them have given ratios a to one another; therefore each side of one of them has b to each side of the other a given ratio.

a 3. def.
b 9. dat.

But if the figure A be not similar to B, let CD, EF be any two of their sides; and upon EF conceive the figure EG to be described similar and similarly placed to the figure A, so that CD, EF be homologous sides; therefore EG is given in species; and the figure B is given in species; wherefore c the ratio of B to EG is given; and the ratio of A to B is given, therefore b the ratio of the figure A to EG is given; and A is similar to EG; therefore d the ratio of the side CD to EF is given; and consequently b the ratios of the remaining sides to the remaining sides are given.
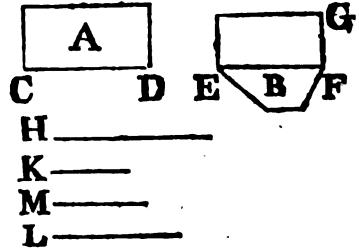
c 53. dat.

d 58. dat.

The ratio of CD to EF may be found thus: Take a straight line H given in magnitude, and because the ratio of the figure A to B is given, make the ratio of H to K the same with it. And by the 53d dat. find the ratio of the figure B to EG, and make the ratio of K to L the same: Between H and L find a mean proportional M, the ratio of CD to EF is the same with the ratio of H to M; because the figure A is to B, as H to K; and as B to EG, so is K to L; ex aequali, as A

to

to EG, fo it H to L : And the figures A, EG are fimilar, and M is a mean proportional between H and L ; therefore, as was fhewn in the preceding propofition, CD is to EF as H to M.

P R O P. LX.

IF a rectilineal figure be given in fpecies and magnitude, the fides of it fhall be given in magnitude.

Let the rectilineal figure A be given in fpecies and magnitude, its fides are given in magnitude.

Take a ftraight line BC given in pofition and magnitude, and upon BC defcribe ᵃ the figure D fimilar, and fimilarly ᵃ 18. 6. placed, to the figure A,
and let EF be the fide of
the figure A homologous
to BC the fide of D ;
therefore the figure D is
given in fpecies. And be-
caufe upon the given
ftraight line BC the fi-
gure D given in fpecies
is defcribed, D is given ᵇ
in magnitude, and the figure A is given in magnitude, there- ᵇ 56. dat.
fore the ratio of A to D is given : And the figure A is fimilar
to D ; therefore the ratio of the fide EF to the homologous fide
BC is given ᶜ ; and BC is given, wherefore ᵈ EF is given : And ᶜ 58. dat.
the ratio of EF to EG is given ᵉ, therefore EG is given. And, ᵈ 2. dat.
in the fame manner, each of the other fides of the figure A can ᵉ 3. def.
be fhewn to be given.
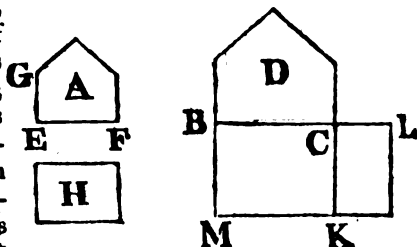
P R O B L E M.

To defcribe a rectilineal figure A fimilar to a given figure D, and equal to another given figure H. It is prop. 25. b. 6. Elem.

Becaufe each of the figures D, H is given, their ratio is gi-
ven, which may be found by making ᶠ upon the given ftraight ᶠ cor. 45.
line BC the parallelogram BK equal to D, and upon its fide ᴵ.
CK making ᶠ the parallelogram KL equal to H in the angle
KCL equal to the angle MBC ; therefore the ratio of D to H,
that is, of BK to KL, is the fame with the ratio of BC to CL :
And becaufe the figures D, A are fimilar, and that the ratio of
D to A, or H, is the fame with the ratio of BC to CL ; by
the 58th dat. the ratio of the homologous fides BC, EF is the
fame with the ratio of BC to the mean proportional between
BC and CL. Find EF the mean proportional ; then EF is the
fide

fide of the figure to be defcribed, homologous to BC the fide of D, and the figure itfelf can be defcribed by the 18th prop. b. 6. which, by the conftruction, is fimilar to D; and becaufe D is to A, as ᵍ BC to CL, that is as the figure BK to KL; and that D is equal to BK, therefore A ʰ is equal to KL, that is, to H.

g 2. Cor.
20. 6.
h 14. 5.

P R O P. LXI.

57.

See N.

IF a parallelogram given in magnitude has one of its fides and one of its angles given in magnitude, the other fide alfo is given.

Let the parallelogram ABDC given in magnitude, have the fide AB and the angle BAC given in magnitude, the other fide AC is given.

Take a ftraight line EF given in pofition and magnitude; and becaufe the parallelogram AD is given in magnitude, a rectilineal figure equal to it can be found ᵃ. And a parallelogram equal to this figure can be applied ᵇ to the given ftraight line EF in an angle equal to the given angle BAC. Let this be the parallelogram EFHG having the angle FEG equal to the angle BAC. And becaufe the parallelograms AD, EH are equal, and have

a 1. def.

b Cor. 45.
1.

the angles at A and E equal; the fides about them are reciprocally proportional ᶜ; therefore as AB to EF, fo is EG to AC; and AB, EF, EG are given, therefore alfo AC is given ᵈ. Whence the way of finding AC is manifeft.

c 14. 6.
d 12. 6.

P R O P. LXII.

H.

See N.
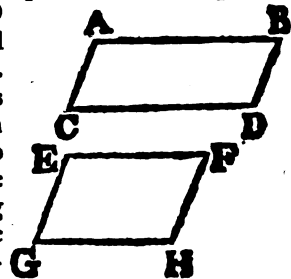
IF a parallelogram has a given angle, the rectangle contained by the fides about that angle has a given ratio to the parallelogram.

Let the parallelogram ABCD have the given angle ABC, the rectangle AB, BC has a given ratio to the parallelogram AC.

From the point A draw AE perpendicular to BC; becaufe the angle ABC is given, as alfo the angle AEB, the triangle ABE is given ᵃ in fpecies; therefore the ratio of BA to AE is given. But as BA to AE, fo is ᵇ the rectangle AB, BC to the rectangle AE, BC; therefore the ratio of

a 43. dat.

b 1. 6.
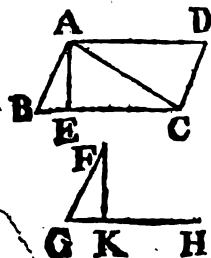
the rectangle AB, BC to AE, BC, that is [c], to the parallelo- [c 35. 1.]
gram AC, is given.

And it is evident how the ratio of the rectangle to the pa-
rallelogram may be found, by making the angle FGH equal
to the given angle ABC, and drawing, from any point F in
one of its sides, FK perpendicular to the other GH ; for GF
is to FK, as BA to AE, that is, as the rectangle AB, BC, to
the parallelogram AC.

Cor. And if a triangle ABC has a given angle ABC, the [66.]
rectangle AB, BC contained by the sides about that angle,
shall have a given ratio to the triangle ABC.

Complete the parallelogram ABCD ; therefore, by this pro-
position, the rectangle AB, BC has a given ratio to the paral-
lelogram AC ; and AC has a given ratio to its half the tri-
angle [d] ABC ; therefore the rectangle AB, BC has a given [e] ra- [d 14. 1.]
tio to the triangle ABC. [e 9. dat.]

And the ratio of the rectangle to the triangle is found thus :
Make the triangle FGK, as was shown in the proposition ; the
ratio of GF to the half of the perpendicular FK is the same with
the ratio of the rectangle AB, BC to the triangle ABC. Be-
cause, as was shown, GF is to FK, as AB, BC to the paralle-
logram AC ; and FK is to its half, as AC is to its half, which
is the triangle ABC ; therefore, ex aequali, GF is to the half of
FK, as AB, BC rectangle is to the triangle ABC.

P R O P. LXIII.

[56.]

IF two parallelograms be equiangular, as a side of the
first to a side of the second, so is the other side of the
second to the straight line to which the other side of the
first has the same ratio which the first parallelogram has
to the second. And consequently, if the ratio of the first
parallelogram to the second be given, the ratio of the
other side of the first to that straight line is given ; and if
the ratio of the other side of the first to that straight line
be given, the ratio of the first parallelogram to the second
is given.

Let AC, DF be two equiangular parallelograms, as BC, a
side of the first, is to EF, a side of the second, so is DE, the o-
ther side of the second, to the straight line to which AB, the o-
ther

ther fide of the firft has the fame ratio which AC has to DF.

Produce the ftraight line AB, and make as BC to EF, fo DE to BG, and complete the parallelo-gram BGHC; therefore, becaufe BC, or GH, is to EF, as DE to BG, the fides about the equal angles BGH, DEF are reciprocally proportional; wherefore ᵃ the parallelogram BH is equal to DF; and AB is to BG, as the parallelogram AC is to BH, that is, to DF; as there-fore BC is to EF, fo is DE to BG; which is the ftraight line to which AB has the fame ratio that AC has to DF.
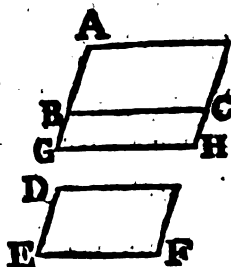
ᵃ 14. 6.

And if the ratio of the parallelogram AC to DF be given, then the ratio of the ftraight line AB to BG is given; and if the ratio of AB to the ftraight line BG be given, the ratio of the parallelogram AC to DF is given.

74. 73.

P R O P. LXIV.

See N.

IF two parallelograms have unequal, but given angles, and if as a fide of the firft to a fide of the fecond, fo the other fide of the fecond be made to a certain ftraight line; if the ratio of the firft parallelogram to the fecond be given, the ratio of the other fide of the firft to that ftraight line fhall be given. And if the ratio of the other fide of the firft to that ftraight line be given, the ratio of the firft parallelogram to the fecond fhall be given.

Let ABCD, EFGH be two parallelograms which have the unequal, but given, angles ABC, EFG; and as BC to FG, fo make EF to the ftraight line M. If the ratio of the parallelo-gram AC to EG be given, the ratio of AB to M is given.

At the point B of the ftraight line BC make the angle CBK equal to the angle EFG, and complete the parallelogram KBCL. And becaufe the ratio of AC to EG is given, and that AC is equal ᵃ to the parallelogram KC, therefore the ratio of KC to EG is given; and KC, EG are equiangular; there-fore as BC to FG, fo is ᵇ EF to the ftraight line to which KB has a given ratio, viz. the fame which the parallelogram KC has to EG: But as BC to FG, fo is EF to the ftraight line M; therefore KB has a given ratio to M; and the ratio

ᵃ 35. 1.

ᵇ 63. dat.

of

of AB to BK is given, becaufe the triangle ABK is given in fpecies ^c; therefore the ratio of AB to M is given ^d.

And if the ratio of AB to M be given, the ratio of the parallelogram AC to EG is given; for fince the ratio of KB to BA is given, as alfo the ratio of AB to M, the ratio of KB to M is given ^d; and becaufe the parallelograms KC, EG are equiangular, as BC to FG, fo is ^b EF to the ftraight line to which KB has the fame ratio which the parallelogram KC has to EG; but as BC to FG, fo is EF to M; therefore KB is to M, as the parallelogram KC is to EG; and

the ratio of KB to M is given, therefore the ratio of the parallelogram KC, that is, of AC to EG, is given.
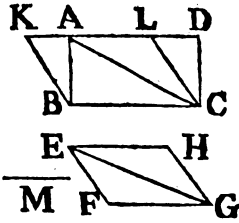
Cor. And if two triangles ABC, EFG have two equal angles, or two unequal, but given, angles ABC, EFG, and if as BC a fide of the firft to FG a fide of the fecond, fo the other fide of the fecond EF be made to a ftraight line M; if the ratio of the triangles be given, the ratio of the other fide of the firft to the ftraight line M is given.

Complete the parallelograms ABCD, EFGH; and becaufe the ratio of the triangle ABC to the triangle EFG is given, the ratio of the parallelogram AC to EG is given ^e, becaufe the parallelograms are double ^f of the triangles; and becaufe BC is to FG, as EF to M, the ratio of AB to M is given by the 63d dat. if the angles ABC, EFG are equal; but if they be unequal, but given angles, the ratio of AB to M is given by this propofition.

And if the ratio of AB to M be given, the ratio of the parallelogram AC to EG is given by the fame propofitions; and therefore the ratio of the triangle ABC to EFG is given.

PROP. LXV.

IF two equiangular parallelograms have a given ratio to one another, and if one fide has to one fide a given ratio; the other fide fhall alfo have to the other fide a given ratio.

Let the two equiangular parallelograms AB, CD have a given ratio to one another, and let the fide EB have a given ratio to the fide FD; the other fide AE has alfo a given ratio to the other fide CF.

D d

Becaufe

Becaufe the two equiangular parallelograms AB, CD have a given ratio to one another; as EB, a fide of the firft, is to FD, a fide of the fecond, fo is ª FC, the other fide of the fecond, to the ftraight line to which AE, the other fide of the firft, has the fame given ratio which the firft parallelogram AB has to the other CD. Let this ftraight line be EG; therefore the ratio of AE to EG is given; and EB is to FD, as FC to EG, therefore the ratio of FC to EG is given, becaufe the ratio of EB to FD is given; and becaufe the ratio of AE to EG, as alfo the ratio of FC to EG is given; the ratio of AE to CF is given ᵇ.

a 63. dat.
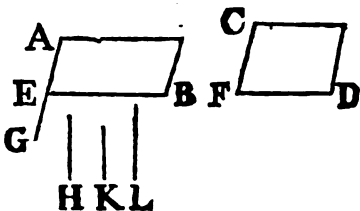
b 9. dat.

The ratio of AE to CF may be found thus: Take a ftraight line H given in magnitude; and becaufe the ratio of the parallelogram AB to CD is given, make the ratio of H to K the fame with it. And becaufe the ratio of FD to EB is given, make the ratio of K to L the fame: The ratio of AE to CF is the fame with the ratio of H to L. Make as EB to FD, fo FC to EG, therefore, by inverfion, as FD to EB, fo is EG to FC; and as AE to EG, fo is ª (the parallelogram AB to CD, and fo is) H to K; but as EG to FC, fo is (FD to EB, and fo is) K to L; therefore, ex aequali, as AE to FC, fo is H to L.

69.

P R O P. LXVI.

IF two parallelograms have unequal, but given angles, and a given ratio to one another; if one fide has to one fide a given ratio, the other fide has alfo a given ratio to the other fide.

Let the two parallelograms ABCD, EFGH which have the given unequal angles ABC, EFG, have a given ratio to one another, and let the ratio of BC to FG be given; the ratio alfo of AB to EF is given.

At the point B of the ftraight line BC make the angle CBK equal to the given angle EFG, and complete the parallelogram BKLC; and becaufe each of the angles BAK, AKB is given, the triangle ABK is given ª in fpecies; therefore the ratio of AB to BK is given; and becaufe, by the hypothefis, the

a 43. dat.

the ratio of the parallelogram AC to EG is given, and that AC is equal [b] to BL; therefore the ratio of BL to EG is given: [b] 35. 1. And becaufe BL is equiangular to EG, and by the hypothefis, the ratio of BC to FG is given; therefore [c] the ratio of KB to [c] 65. dat. EF is given, and the ratio of KB to BA is given; the ratio there-fore [d] of AB to EF is given.

[d] 9. dat.
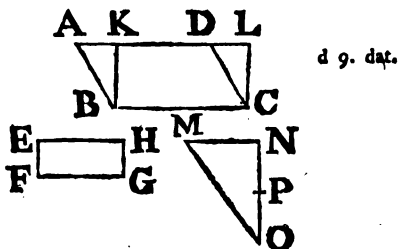
The ratio of AB to EF may be found thus: Take the ftraight line MN given in pofition and magni-tude; and make the angle NMO equal to the given angle BAK, and the angle MNO equal to the given angle EFG or AKB: And becaufe the parallelogram BL is equiangular to EG, and has a given ratio to it, and that the ratio of BC to FG is given; find by the 65th dat. the ratio of KB to EF; and make the ratio of NO to OP the fame with it: Then the ratio of AB to EF is the fame with the ratio of MO to OP: For fince the triangle ABK is equiangular to MON, as AB to BK, fo is MO to ON; and as KB to EF, fo is NO to OP; therefore, ex aequali, as AB to EF, fo is MO to OP.

P R O P. LXVII. 70.

IF the fides of two equiangular parallelograms have given ratios to one another; the parallelograms fhall have a given ratio to one another.

See N.

Let ABCD, EFGH be two equiangular parallelograms, and let the ratio of AB to EF, as alfo the ratio of BC to FG, be gi-ven; the ratio of the parallelogram AC to EG is given.

Take a ftraight line K given in magnitude, and becaufe the ratio of AB to EF is given, make the ratio of K to L the fame with it; therefore L is given [a]: And becaufe the ratio of BC to FG is given, make the ratio of L to M the fame: Therefore M is given [a]; and K is given, wherefore [b] the ratio of K to M is given: But the parallelogram AC is to the parallelogram EG, as the ftraight line K to the ftraight line M,

[a] 2. dat.

[b] 1. dat.

D d 2 as
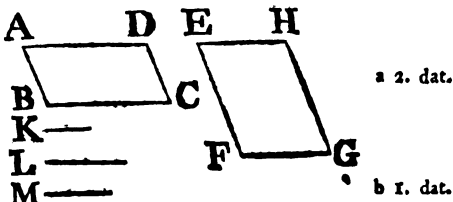
as is demonftrated in the 23d prop. of b. 6. Elem. therefore the ratio of AC to EG is given.

From this it is plain how the ratio of two equiangular parallelograms may be found when the ratios of their fides are given.

78.

P R O P. LXVIII.

See N.

IF the fides of two parallelograms which have unequal, but given angles, have given ratios to one another; the parallelograms fhall have a given ratio to one another.

Let two parallelograms ABCD, EFGH which have the given unequal angles ABC, EFG have the ratios of their fides, viz. of AB to EF, and of BC to FG, given; the ratio of the parallelogram AC to EG is given.

At the point B of the ftraight line BC make the angle CBK equal to the given angle EFG, and complete the parallelogram KBCL: And becaufe each of the angles BAK, BKA is

a 43. dat. given, the triangle ABK is given a in fpecies: Therefore the ratio of AB to BK is given; and the ratio of AB to EF is gi

b 9. dat. ven, wherefore b the ratio of BK to EF is given: And the ratio of BC to FG is given;
and the angle KBC is equal
to the angle EFG; there

c 67. dat. fore c the ratio of the parallelogram KC to EG is gi

d 35. 1. ven: But KC is equal d to
AC; therefore the ratio of
AC to EG is given.
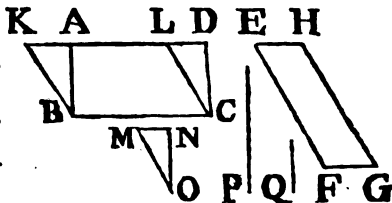
The ratio of the parallelogram AC to EG may be found thus: Take the ftraight line MN given in pofition and magnitude, and make the angle MNO equal to the given angle KAB, and the angle NMO equal to the given angle AKB or FEH: And becaufe the ratio of AB to EF is given, make the ratio of NO to P the fame; alfo make the ratio of P to Q the fame with the given ratio of BC to FG, the parallelogram AC is to EG, as MO to Q.

Becaufe the angle KAB is equal to the angle MNO, and the angle AKB equal to the angle NMO; the triangle AKB is equiangular to NMO: Therefore as KB to BA, fo is MO to ON; and as BA to EF, fo is NO to P; wherefore, ex aequali, as KB to EF, fo is MO to P: And BC is to FG, as P

to

to Q, and the parallelograms KC, EG are equiangular; therefore, as was ſhown in prop. 67. the parallelogram KC, that is, AC, is to EG, as MO to Q.

Cor. 1. If two triangles ABC, DEF have two equal angles, or two unequal, but given angles ABC, DEF, and if the ratios of the ſides about theſe angles, viz. the ratios of AB to DE, and of BC to EF be given; the triangles ſhall have a given ratio to one another.
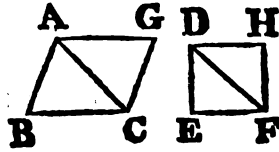
Complete the parallelograms BG, EH; the ratio of BG to EH is given [a]; and therefore the triangles which are the halves [b] of them have a given [c] ratio to one another.

Cor. 2. If the baſes BC, EF of two triangles ABC, DEF have a given ratio to one another, and if alſo the ſtraight lines AG, DH which are drawn to the baſes from the oppoſite angles, either in equal angles, or unequal, but given angles AGC, DHF have a given ratio to one another; the triangles ſhall have a given ratio to one another.

Draw BK, EL parallel to AG, DH, and complete the parallelograms KC, LF. And becauſe the angles AGC, DHF, or their equals, the angles KBC, LEF are either equal, or unequal, but given; and that the ratio of AG to DH, that is, of KB to LE, is given, as alſo the ratio of BC to EF; therefore [a] the ratio of the parallelogram KC to LF is given; wherefore alſo the ratio of the triangle ABC to DEF is given [b].

71.

a 67. or 68. dat.
b 34. 1.
c 15. 5.

72.

a 67. or 68. dat.
b { 41. 1. 15. 5.

P R O P. LXIX.

61.

IF a parallelogram which has a given angle be applied to one ſide of a rectilineal figure given in ſpecies; if the figure have a given ratio to the parallelogram, the parallelogram is given in ſpecies.

Let ABCD be a rectilineal figure given in ſpecies, and to one ſide of it AB, let the parallelogram ABEF having the given angle ABE be applied; if the figure ABCD has a given ratio to the parallelogram BF, the parallelogram BF is given in ſpecies.

Through the point A draw AG parallel to BC, and through the point C draw CG parallel to AB, and produce GA, CB to

the points H, K; becauſe the angle ABC is given [a], and the ratio of AB to BC is given, the figure ABCD being given in ſpecies; therefore, the parallelogram BG is given [a] in ſpecies. And becauſe upon the ſame ſtraight line AB the two rectilineal figures BD, BG given in ſpecies are deſcribed, the ratio of BD to BG is given [b]; and, by hypotheſis, the ratio of BD to the parallelogram BF is given; wherefore [c] the ratio of BF, that is [d], of the parallelogram BH, to BG is given, and therefore [e] the ratio of the ſtraight line KB to BC is given; and the ratio of BC to BA is given, wherefore the ratio of KB to BA is given [c]: And becauſe the angle ABC is given, the adjacent angle ABK is given; and the angle ABE is given, therefore the remaining angle KBE is given. The angle EKB is alſo given, becauſe it is equal to the angle ABK; therefore the triangle BKE is given in ſpecies, and conſequently the ratio of EB to BK is given; and the ratio of KB to BA is given, wherefore [c] the ratio of EB to BA is given; and the angle ABE is given, therefore the parallelogram BF is given [a] in ſpecies.

A parallelogram ſimilar to BF may be found thus: Take a ſtraight line LM given in poſition and

magnitude; and becauſe the angles ABK, ABE are given, make the angle NLM equal to ABK, and the angle NLO equal to ABE. And becauſe the ratio of BF to BD is given, make the ratio of LM to P the ſame with it; and becauſe the ratio of the figure BD to BG is given, find this ratio by the 53d dat. and make the ratio of P to Q the ſame. Alſo, becauſe the ratio of CB to BA is given, make the ratio of Q to R the ſame; and take LN equal to R; through the point M draw OM parallel to LN, and complete the parallelogram NLOS; then this is ſimilar to the parallelogram BF.
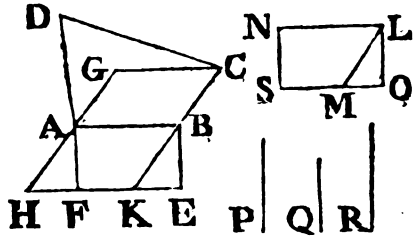
Becauſe the angle ABK is equal to NLM, and the angle ABE to NLO the angle KBE is equal to MLO; and the angles BKE, LMO are equal, becauſe the angle ABK is e-qual to NLM; therefore, the triangles BKE, LMO are equi-angular to one another; wherefore as BE to BK, ſo is LO to LM; and becauſe as the figure BF to BD, ſo is the ſtraight line LM to P; and as BD to BG, ſo is P to Q; ex aequali, as BF, that is [d] BH, to BG, ſo is LM to Q: But BH is to [c] BG,

BG, as KB to BC; as therefore KB to BC, fo is LM to Q;
and becaufe BE is to BK as LO to LM; and as BK to BC,
fo is LM to Q: And as BC to BA, fo Q was made to R; there-
fore, ex aequali, as BE to BA, fo is LO to R, that is to LN;
and the angles ABE, NLO are equal; therefore the paralle-
logram BF is fimilar to LS.

P R O P. LXX.

IF two ftraight lines have a given ratio to one another, See N.
and upon one of them be defcribed a rectilineal figure
given in fpecies, and upon the other a parallelogram ha-
ving a given angle; if the figure have a given ratio to
the parallelogram, the parallelogram is given in fpecies.

Let the two ftraight lines AB, CD have a given ratio to one
another, and upon AB let the figure AEB given in fpecies be
defcribed, and upon CD the parallelogram DF having the given
angle FCD; if the ratio of AEB to DF be given, the paralle-
logram DF is given in fpecies.

Upon the ftraight line AB, conceive the parallelogram AG
to be defcribed fimilar, and fimilarly placed to FD; and becaufe
the ratio of AB to CD is given, and upon them are defcribed
the fimilar rectilineal figures AG,
FD; the ratio of AG to FD is gi-
ven [a]; and the ratio of FD to AEB
is given; therefore [b] the ratio of
AEB to AG is given; and the angle
ABG is given, becaufe it is equal to
the angle FCD; becaufe therefore
the parallelogram AG which has a
given angle ABG is applied to a fide
AB of the figure AEB given in fpe-

a 54. dat.
b 9. dat.

cies, and the ratio of AEB to AG is given, the parallelogram
AG is given [c] in fpecies; but FD is fimilar to AG; therefore c 69. dat.
FD is given in fpecies.
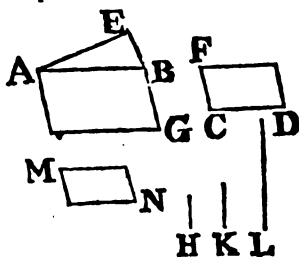
A parallelogram fimilar to FD may be found thus: Take a
ftraight line H given in magnitude; and becaufe the ratio of
the figure AEB to FD is given, make the ratio of H to K the
fame with it: Alfo, becaufe the ratio of the ftraight line CD to
AB is given, find by the 54th dat. the ratio which the figure
FD defcribed upon CD has to the figure AG defcribed upon
AB fimilar to FD; and make the ratio of K to L the fame
with this ratio: And becaufe the ratios of H to K, and of K

to

to L are given, the ratio of H to L is given **; becaufe, there,
fore, as AEB to FD, fo is H to K; and as FD to AG, fo is
K to L; ex aequali, as AEB to AG, fo is H to L; there-
fore the ratio of AEB to AG is given; and the figure AEB
is given in fpecies, and to its fide AB the parallelogram AG
is applied in the given angle ABG; therefore by the 69th dat.
a parallelogram may be found fimilar to AG: Let this be the
parallelogram MN; MN alfo is fimilar to FD; for, by the con-
ftruction, MN is fimilar to ΛG, and AG is fimilar to FD;
therefore the parallelogram FD is fimilar to MN.

P R O P. LXXI.

IF the extremes of three proportional ftraight lines have
given ratios to the extremes of other three propor-
tional ftraight lines; the means fhall alfo have a given
ratio to one another: And if one extreme has a given ra-
tio to one extreme, and the mean to the mean; likewife
the other extreme fhall have to the other a given ratio.

Let A, B, C be three proportional ftraight lines, and D, E,
F three other; and let the ratios of A to D, and of C to F be
given; then the ratio of B to E is alfo given.

Becaufe the ratio of A to D, as alfo of C to F is given, the
ratio of the rectangle A, C to the rectangle D, F is given a;
but the fquare of B is equal b to the rectangle A, C; and the
fquare of E to the rectangle b D, F; therefore the ratio of the
fquare of B to the fquare of E is given; wherefore c alfo the ra-
tio of the ftraight line B to E is given.

Next, let the ratio of A to D, and of B to E be gi-
ven; then the ratio of C to F is alfo given.

Becaufe the ratio of B to E is given, the ratio of
the fquare of B to the fquare of E is given d; there-
fore b the ratio of the rectangle A, C to the rectangle
D, F is given; and the ratio of the fide A to the fide
D is given; therefore the ratio of the other fide C to
the other F is given e.

A B C
D E F

Cor. And if the extremes of four proportionals have to the
extremes of four other proportionals given ratios, and one of
the means a given ratio to one of the means; the other mean
fhall have a given ratio to the other mean, as may be fhown in
the fame manner as in the foregoing propofition.

                                        P R O P.

P R O P. LXXII. 82.

IF four ftraight lines be proportionals ; as the firft is to the ftraight line to which the fecond has a given ratio, fo is the third to a ftraight line to which the fourth has a given ratio.

Let A, B, C, D be four proportional ftraight lines, viz. as A to B, fo C to D ; as A is to the ftraight line to which B has a given ratio, fo is C to a ftraight line to which D has a given ratio.

Let E be the ftraight line to which B has a given ratio, and as B to E, fo make D to F : The ratio of B to E is given [a], and therefore the ratio of D to F ; and becaufe as A to B, fo is C to D ; and as B to E fo D to F ; therefore, ex aequali, as A to E, fo is C to F ; and E is the ftraight line to which B has a given ratio, and F that to which D has a given ratio ; therefore as A is to the ftraight line to which B has a given ratio, fo is C to a line to which D has a given ratio.

a Hyp.

A B E
C D F
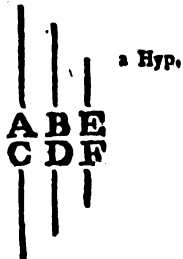
P R O P. LXXIII. 83.

IF four ftraight lines be proportionals ; as the firft is to the ftraight line to which the fecond has a given ratio, fo is a ftraight line to which the third has a given ratio to the fourth.

See N.

Let the ftraight line A be to B, as C to D ; as A to the ftraight line to which B has a given ratio, fo is a ftraight line to which C has a given ratio to D.

Let E be the ftraight line to which B has a given ratio, and as B to E, fo make F to C ; becaufe the ratio of B to E is given, the ratio of C to F is given : And becaufe A is to B, as C to D ; and as B to E, fo F to C ; therefore, ex aequali in proportione perturbata [a], A is to E, as F to D ; that is, A is to E to which B has a given ratio, as F, to which C has a given ratio, is to D.

a 23. 5.

A B E
F C D

P R O P.

64.   P R O P. LXXIV.

IF a triangle has a given obtuse angle ; the excess of
the square of the side which subtends the obtuse
angle, above the squares of the sides which contain it,
shall have a given ratio to the triangle.

Let the triangle ABC have a given obtuse angle ABC; and
produce the straight line CB, and from the point A draw
AD perpendicular to BC : The excess of the square of AC a-
bove the squares of AB, BC, that is [a], the double of the rect-
angle contained by DB, BC, has a given ratio to the triangle
ABC.

.Because the angle ABC is given, the angle ABD is also gi-
ven ; and the angle ADB is given ; wherefore the triangle
ABD is given [b] in species ; and therefore the ratio of AD to
DB is given : And as AD to DB, so is [c] the rectangle AD,
BC to the rectangle DB, BC ; wherefore the ratio of the rectangle
AD, BC to the rectangle DB, BC is given, as also the ratio of twice
the rectangle DB, BC to the rectangle AD,
BC : But the ratio of the rectangle AD,
BC to the triangle ABC is given, because
it is double [d] of the triangle; therefore
the ratio of twice the rectangle DB, BC
to the triangle ABC is given [e] ; and twice
the rectangle DB, BC is the excess [a] of
the square of AC above the squares of AB, BC ; therefore this
excess has a given ratio to the triangle ABC.
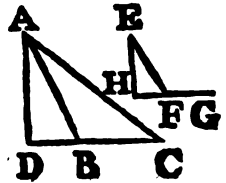
And the ratio of this excess to the triangle ABC may be
found thus : Take a straight line EF given in position and mag-
nitude ; and because the angle ABC is given, at the point F
of the straight line EF, make the angle EFG equal to the angle
ABC ; produce GF, and draw EH perpendicular to FG ; then
the ratio of the excess of the square of AC above the squares
of AB, BC to the triangle ABC, is the same with the ratio of
quadruple the straight line HF to HE.

Because the angle ABD is equal to the angle EFH, and
the angle ADB to EHF, each being a right angle ; the tri-
angle ADB is equiangular to EHF ; therefore [f] as BD to DA,
so FH to HE ; and as quadruple of BD to DA, so is [g] qua-
druple of FH to HE : But as twice BD is to DA, so is [c] twice
the rectangle DB, BC to the rectangle AD, BC ; and as DA
to the half of it, so is [h] the rectangle AD, BC to its half the
triangle

a 12. 2.

b 43. dat.
c 1. 6.

d 41. 1.

e 9. dat.

f 4. 6.
g Cor. 4. 5.

h C. 5.

triangle ABC; therefore, ex aequali, as twice BD is to the half of DA, that is, as quadruple of BD is to DA, that is, as quadruple of FH to HE, so is twice the rectangle DB, BC to the triangle ABC.

P R O P. LXXV.

65.

IF a triangle has a given acute angle, the space by which the square of the side subtending the acute angle is less than the squares of the sides which contain it, shall have a given ratio to the triangle.

Let the triangle ABC have a given acute angle ABC, and draw AD perpendicular to BC, the space by which the square of AC is less than the squares of AB, BC, that is [a], the double of the rectangle contained by CB, BD, has a given ratio to the triangle ABC.

a 13. 2.

Because the angles ABD, ADB are each of them given, the triangle ABD is given in species; and therefore the ratio of BD to DA is given : And as BD to DA, so is the rectangle CB, BD to the rectangle CB, AD; therefore the ratio of these rectangles is given, as also the ratio of twice the rectangle CB, BD to the rectangle CB, AD; but the rectangle CB, AD has a given ratio to its half the triangle ABC; therefore [b] the ratio of twice the rectangle CB, BD to the triangle ABC is given; and twice the rectangle CB, BD is [a] the space by which the square of AC is less than the squares of AB, BC; therefore the ratio of this space to the triangle ABC is given : And the ratio may be found as in the preceding proposition.
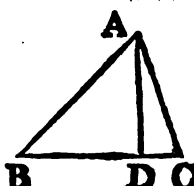
b 9. dat.

L E M M A.

IF from the vertex A of an isosceles triangle ABC, any straight line AD be drawn to the base BC, the square of the side AB is equal to the rectangle BD, DC of the segments of the base together with the square of AD; but if AD be drawn to the base produced, the square of AD is equal to the rectangle BD, DC together with the square of AB.

CAS. 1. Bisect the base BC in E, and join AE which will be perpendicular [a] to BC; wherefore the square of AB is equal [b] to the squares of AE, EB; but the square of EB is equal [c] to the rectangle BD, DC together with the square of DE; therefore the square of AB is equal to the
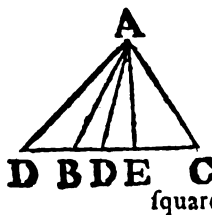
a 8. 1.

b 4. 7. 1.
c 5. 1.

squares

b 47. 1.

fquares of AE, ED, that is, to **b** the fquare of AD, together
with the rectangle BD, DC; the other cafe is fhown in the fame
way by 6. 2. Elem.

67.

P R O P. LXXVI.

IF a triangle have a given angle, the excefs of the fquare
of the ftraight line which is equal to the two fides that
contain the given angle, above the fquare of the third fide,
fhall have a given ratio to the triangle.

Let the triangle ABC have the given angle BAC, the excefs
of the fquare of the ftraight line which is equal to BA, AC to-
gether above the fquare of BC, fhall have a given ratio to the
triangle ABC.

Produce BA, and take AD equal to AC, join DC and
produce it to E, and through the point B draw BE parallel to
AC; join AE, and draw AF perpendicular to DC; and be-
caufe AD is equal to AC, BD is equal to BE; and BC is
drawn from the vertex B of the ifofceles triangle DBE, there-
fore, by the Lemma, the fquare of BD, that is, of BA and
AC together, is equal to the rectangle DC, CE together with
the fquare of BC; and, therefore, the fquare of BA, AC to-
gether, that is, of BD, is greater than
the fquare of BC by the rectangle DC,
CE; and this rectangle has a given
ratio to the triangle ABC: Becaufe
the angle BAC is given, the adjacent
angle CAD is given; and each of the
angles ADC, DCA is given, for each

a 5. & 32. 1. of them is the half **a** of the given angle
BAC; therefore the triangle ADC is

b 43. dat. given **b** in fpecies; and AF is drawn
from its vertex to the bafe in a given angle; wherefore the ratio

c 50. dat. of AF to the bafe CD is given **c**; and as CD to AF, fo is **d** the
d 1. 6. rectangle DC, CE to the rectangle AF, CE; and the ratio of

e 41. 1. the rectangle AF, CE to its half **e** the triangle ACE is given;
therefore the ratio of the rectangle DC, CE to the triangle

f 37. 1. ACE, that is **f**, to the triangle ABC, is given **s**; and the rectangle
s 9. dat. DC, CE is the excefs of the fquare of BA, AC together above
the fquare of BC; therefore the ratio of this excefs to the tri-
angle ABC is given.
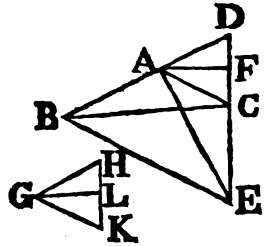
The ratio which the rectangle DC, CE has to the triangle
ABC is found thus: Take the ftraight line GH given in pofi-

tion and magnitude, and at the point G in GH make the angle
HGK equal to the given angle CAD, and take GK equal to
GH, join KH, and draw GL perpendicular to it: Then the
ratio of HK to the half of GL is the same with the ratio of
the rectangle DC, CE to the triangle ABC : Because the angles
HGK, DAC at the vertices of the isosceles triangles GHK,
ADC are equal to one another, these triangles are similar ; and
because GL, AF are perpendicular to the bases HK, DC, as ⎱ h 4. 6.
HK to GL, so is [h] (DC to AF, and so is) the rectangle DC, ⎰ h 22. 5.
CE to the rectangle AF, CE ; but as GL to its half, so is the
rectangle AF, CE to its half, which is the triangle ACE, or
the triangle ABC ; therefore, ex aequali, HK is to the half of
the straight line GL, as the rectangle DC, CE is to the triangle
ABC.

Cor. And if a triangle have a given angle, the space by
which the square of the straight line which is the difference of
the sides which contain the given angle is less than the square
of the third side, shall have a given ratio to the triangle. This
is demonstrated the same way as the preceding proposition, by
help of the second case of the Lemma.

PROP. LXXVII.   1.

IF the perpendicular drawn from a given angle of a See N.
   triangle to the opposite side, or base, has a given ra-
tio to the base, the triangle is given in species.

Let the triangle ABC have the given angle BAC, and let the
the perpendicular AD drawn to the base BC, have a given ratio
to it, the triangle ABC is given in species.
If ABC be an isosceles triangle, it is evident [a] that if any a 5. & 32.1.
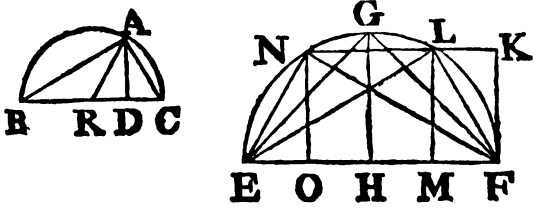
one of its angles be given, the rest are also given ; and there-
fore the triangle is given in species, without the consideration
of the ratio of the perpendicular to the base, which in this case
is given by prop. 50.
But when ABC is not an isosceles triangle, take any straight
line EF given in position and magnitude, and upon it describe
        the

the fegment of a circle EGF containing an angle equal to the
given angle BAC, draw GH bifecting EF at right angles, and
join EG, GF : Then, fince the angle EGF is equal to the angle
BAC, and that EGF is an ifofceles triangle, and ABC is not,
the angle FEG is not equal to the angle CBA : Draw EL ma-
king the angle FEL equal to the angle CBA ; join FL, and
draw LM perpendicular to EF; then, becaufe the triangles ELF,
BAC are equiangular, as alfo are the triangles MLE, DAB,
as ML to LE, fo is DA to AB; and as LE to EF, fo is AB to BC;
wherefore, ex aequali, as LM to EF, fo is AD to BC ; and be-
caufe the ratio of AD to BC is given, therefore the ratio of LM

b 2. dat.   to EF is given ; and EF is given, wherefore b LM alfo is given.
Complete the parallelogram LMFK; and becaufe LM is given,
FK is given in magnitude; it is alfo given in pofition, and
c 30. dat.   the point F is given, and confequently c the point K; and be-
caufe through K the ftraight line KL is drawn parallel to EF
d 31. dat.   which is given in pofition, therefore d KL is given in pofition;
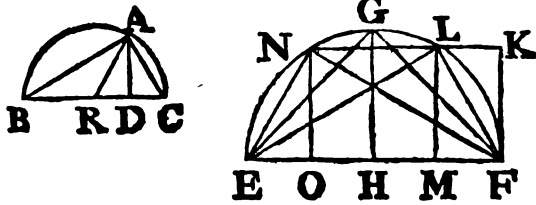
and the circumference ELF is given in pofition ; therefore the
e 28. dat.   point L is given e.   And becaufe the points L, E, F are given,
f 29. dat.   the ftraight lines LE, EF, FL are given f in magnitude ; there-
g 42. dat.   fore the triangle LEF is given in fpecies g ; and the triangle
ABC is fimilar to LEF, wherefore alfo ABC is given in fpecies.

Becaufe LM is lefs than GH, the ratio of LM to EF, that
is, the given ratio of AD to BC, muft be lefs than the ratio of
GH to EF, which the ftraight line, in a fegment of a circle con-
taining an angle equal to the given angle, that bifects the bafe
of the fegment at right angles, has unto the bafe.

Cor. 1. If two triangles, ABC, LEF have one angle BAC
equal to one angle ELF, and if the perpendicular AD be to the
bafe BC, as the perpendicular LM to the bafe EF, the triangles
ABC, LEF are fimilar.

Defcribe the circle EGF about the triangle ELF, and draw
LN parallel to EF, join EN, NF, and draw NO perpendicu-
lar to EF; becaufe the angles ENF, ELF are equal, and that
the

the angle EFN is equal to the alternate angle FNL, that is, to the angle FEL in the same segment; therefore the triangle NEF is similar to LEF; and in the segment EGF there can be no other triangle upon the base EF, which has the ratio of its perpendicular to that base the same with the ratio of LM or NO to EF, because the perpendicular must be greater or less than LM or NO; but, as has been shewn in the preceding demonstration, a triangle similar to ABC can be described in the segment EGF upon the base EF, and the ratio of its perpendicular to the base is the same, as was there shewn, with the ratio of AD to BC, that is, of LM to EF; therefore that triangle must be either LEF, or NEF, which therefore are similar to the triangle ABC.

Cor. 2. If a triangle ABC has a given angle BAC, and if the straight line AR drawn from the given angle to the opposite side BC, in a given angle ARC, has a given ratio to BC, the triangle ABC is given in species.

Draw AD perpendicular to BC; therefore the triangle ARD is given in species; wherefore the ratio of AD to AR is given; and the ratio of AR to BC is given, and consequently [h] the ratio of AD to BC is given; and the triangle ABC is therefore given in species [i].   h 9. dat.
    i 77. dat.

Cor. 3. If two triangles ABC, LEF have one angle BAC equal to one angle ELF, and if straight lines drawn from these angles to the bases, making with them given and equal angles, have the same ratio to the bases, each to each; then the triangles are similar; for having drawn perpendiculars to the bases from the equal angles, as one perpendicular is to its base, so is the other to its base [k]; wherefore, by Cor. 1. the triangles are similar.   k { 4. 6.
    22. 5.

A triangle similar to ABC may be found thus: Having described the segment EGF and drawn the straight line GH as was directed in the proposition, find FK which has to EF the given ratio of AD to BC; and place FK at right angles to EF from the point F; then because, as has been shewn, the ratio of AD to BC, that is, of FK to EF, must be less than the ratio of GH to EF; therefore FK is less than GH; and consequently the parallel to EF drawn through the point K, must meet the circumference of the segment in two points: Let L be either of them, and join EL, LF, and draw LM perpendicular to EF; then, because the angle BAC is equal to the angle ELF, and that AD is to BC, as KF, that is, LM to EF, the triangle ABC is similar to the triangle LEF, by Cor. 1.

P R O P.

P R O P. LXXVIII.

IF a triangle have one angle given, and if the ratio of
the rectangle of the fides which contain the given angle
to the fquare of the third fide be given, the triangle is
given in fpecies.

Let the triangle ABC have the given angle BAC, and let the
ratio of the rectangle BA, AC to the fquare of BC be given;
the triangle ABC is given in fpecies.

From the point A, draw AD perpendicular to BC, the rect-
angle AD, BC has a given ratio to its half **a** the triangle ABC;
and becaufe the angle BAC is given, the ratio of the triangle
ABC to the rectangle BA, AC is given **b**; and, by the hypo-
thefis, the ratio of the rectangle BA, AC to the fquare of BC is
given; therefore **c** the ratio of the rectangle AD, BC to the
fquare of BC, that is, **d** the ratio of the ftraight line AD to BC,
is given; wherefore the triangle ABC is given in fpecies **e**.

A triangle fimilar to ABC may be found thus: Take a
ftraight line EF given in pofition and magnitude, and make
the angle FEG equal to the given angle BAC, and draw FH
perpendicular to EG, and BK perpendicular to AC; therefore
the triangles ABK, EFH
are fimilar, and the rect
angle AD, BC, or the
rectangle BK, AC, which
is equal to it, is to the
rectangle BA, AC, as the
ftraight line BK to BA,
that is, as FH to FE. Let
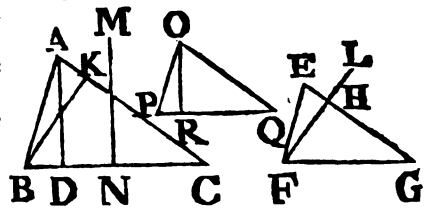
the given ratio of the rectangle BA, AC to the fquare of BC
be the fame with the ratio of the ftraight line EF to FL; there-
fore, ex aequali, the ratio of the rectangle AD, BC to the
fquare of BC, that is, the ratio of the ftraight line AD to BC,
is the fame with the ratio of HF to FL; and becaufe AD is
not greater than the ftraight line MN in the fegment of the
circle defcribed about the triangle ABC, which bifects BC
at right angles; the ratio of AD to BC, that is, of HF
to FL, muft not be greater than the ratio of MN to BC:
Let it be fo, and, by the 77th dat find a triangle OPQ
which has one of its angles POQ equal to the given
angle BAC, and the ratio of the perpendicular OR,
drawn from that angle to the bafe PQ the fame with the
ratio of HF to FL; then the triangle ABC is fimilar to
                                                              OPQ

Margin notes:
a 41. 1.
b Cor. 62. dat.
c 9. dat.
d 1. 6.
e 77. dat.

OPQ: Becaufe, as has been fhown, the ratio of AD to BC is the fame with the ratio of (HF to FL, that is, by the conftruction, with the ratio of) OR to PQ; and the angle BAC is equal to the angle POQ; therefore the triangle ABC is fimilar f to the triangle POQ.

f 1. Cor. 77. Dat.

Otherwife,

Let the triangle ABC have the given angle BAC, and let the ratio of the rectangle BA, AC to the fquare of BC be given; the triangle ABC is given in fpecies.

Becaufe the angle BAC is given, the excefs of the fquare of both the fides BA, AC together above the fquare of the third fide BC has a given a ratio to the triangle ABC. Let the figure D be equal to this excefs; therefore the ratio of D to the triangle ABC is given; and the ratio of the triangle ABC to the rectangle BA, AC is given b, becaufe BAC is a given angle; and the rectangle BA, AC has a given ratio to the fquare of BC; wherefore c the ratio of D to the fquare of BC is given; and, by compofition d, the ratio of the fpace D together with the fquare of BC to the fquare of BC is given; but D together with the fquare of BC is equal to the fquare of both BA and AC together; therefore the ratio of the fquare of BA, AC together to the fquare of BC is given; and the ratio of BA, AC together to BC is therefore given e; and the angle BAC is given, wherefore f the triangle ABC is given in fpecies.

a 76. dat.

b Cor. 6. dat.

c 10. dat.

d 7. dat.

e 59. dat.

f 48. dat.
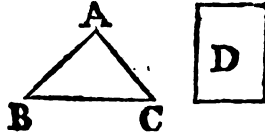
The compofition of this, which depends upon thofe of the 76th and 48th propofitions, is more complex than the preceding compofition, which depends upon that of prop. 77. which is eafy.

P R O P. LXXIX.   K.

IF a triangle have a given angle, and if the ftraight line See N. drawn from that angle to the bafe, making a given angle with it, divides the bafe into fegments which have a given ratio to one another; the triangle is given in fpecies.

Let the triangle ABC have the given angle BAC, and let the ftraight line AD drawn to the bafe BC making the given angle ADB, divide BC into the fegments BD, DC which have

E f   a

a given ratio to one another; the triangle ABC is given in fpecies.

Defcribe ᵃ the circle BAC about the triangle, and from its centre E, draw EA, EB, EC, ED; becaufe the angle BAC is
given, the angle BEC at the centre, which is the double ᵇ of it, is given. And the ratio of BE to EC is given, becaufe they
are equal to one another; therefore ᶜ the triangle BEC is given in fpecies, and the ratio of EB to BC given; alfo the
ratio of CB to BD is given ᵈ, becaufe the ratio of BD to DC
is given; therefore the ratio of EB to BD is given ᵉ, and the angle EBC is given, wherefore the triangle EBD is given ᶜ in fpecies, and the ratio of EB, that is, of EA to ED, is therefore given; and the angle EDA is given, becaufe each of the angles BDE, BDA is given; therefore the triangle AED is
given ᶠ in fpecies, and the angle AED given; alfo the angle DEC is given, becaufe each of the angles BED, BEC is given; therefore the angle AEC is given, and the ratio of EA to EC, which are equal, is given; and the triangle AEC is therefore given ᵉ in fpecies, and the angle ECA given; and the angle ECB is given, wherefore the angle ACB is given, and the angle BAC is alfo
given; therefore ᵍ the triangle ABC is given in fpecies.
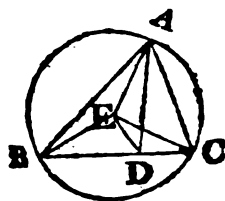
A triangle fimilar to ABC may be found, by taking a ftraight line given in pofition and magnitude, and dividing it in the given ratio which the fegments BD, DC are required to have to one another; then, if upon that ftraight line a fegment of a circle be defcribed containing an angle equal to the given angle BAC, and a ftraight line be drawn from the point of divifion in an angle equal to the given angle ADB, and from the point where it meets the circumference, ftraight lines be drawn to the extremity of the firft line, thefe, together with the firft line, fhall contain a triangle fimilar to ABC, as may eafily be fhown.

The demonftration may be alfo made in the manner of that of the 77th prop. and that of the 77th may be made in the manner of this.

L.

P R O P. LXXX.

IF the fides about an angle of a triangle have a given ratio to one another, and if the perpendicular drawn from that angle to the bafe has a given ratio to the bafe; the triangle is given in fpecies.

Let

Let the fides BA, AC, about the angle BAC of the triangle
ABC have a given ratio to one another, and let the perpendicu-
lar AD have a given ratio to the bafe BC ; the triangle ABC is
given in fpecies.

First, let the fides AB, AC be equal to one another, there-
fore the perpendicular AD bifects[a] the bafe
BC ; and the ratio of AD to BC, and there-
fore to its half DB, is given ; and the angle
ADB is given ; wherefore the triangle [*] ABD,
and confequently the triangle ABC, is given[b]
in fpecies.

a 26. 1.
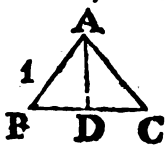
[*] 43. dat.

b 44. dat.

But let the fides be unequal, and BA be greater than AC ;
and make the angle CAE equal to the angle ABC ; becaufe
the angle AEB is common to the triangles AEB, CEA, they
are fimilar ; therefore as AB to BE, fo is CA to AE, and, by
permutation, as BA to AC, fo is BE to EA, and fo is EA
to EC ; and the ratio of BA to AC is given, therefore the
ratio of BE to EA, and the ratio of EA to EC, as alfo the
ratio of BE to EC is given[c] ; wherefore the ratio of EB to
BC is given[d] ; and the ratio of AD to BC
is given by the hypothefis, therefore[c] the
ratio of AD to BE is given ; and the ratio
of BE to EA was fhown to be given ; where-
fore the ratio of AD to AE is given, and
ADE is a right angle, therefore the triangle
ADE is given[e] in fpecies, and the angle AEB given ; the ra-
tio of BE to EA is likewife given, therefore[b] the triangle ABE
is given in fpecies, and confequently the angle EAB, as alfo
the angle ABE, that is, the angle CAE, is given ; therefore the
angle BAC is given, and the angle ABC being alfo given, the
triangle ABC is given[f] in fpecies.

c 9. dat.

d 6. dat.

e 46. dat.

f 43. dat.

How to find a triangle which fhall have the things which
are mentioned to be given in the propofition, is evident in
the firft cafe ; and to find it the more eafily in the other
cafe, it is to be obferved that, if the ftraight line EF equal to
EA be placed in EB towards B, the point F divides the bafe
BC into the fegments BF, FC which have to one another the
ratio of the fides BA, AC ; becaufe BE, EA, or EF, and
EC were fhown to be proportionals, therefore [*] BF is to FC,
as BE to EF, or EA, that is, as BA to AC ; and AE cannot
be lefs than the altitude of the triangle ABC, but it may be

[*] 19. 5.

E e 2

equal

equal to it; which, if it be, the triangle, in this cafe, as alfo
the ratio of the fides, may be thus found; having given the
ratio of the perpendicular to the bafe. Take the ftraight line
GH given in pofition and magnitude, for the bafe of the tri-
angle to be found; and let the given ratio of the perpendicu-
lar to the bafe be that of the ftraight line K to GH, that is,
let K be equal to the perpendicular; and fuppofe GLH to be
the triangle which is to be found, therefore having made the
angle HLM equal to LGH, it is required that LM be per-
pendicular to GM, and equal to K; and becaufe GM, ML,
MH are proportionals, as was fhown of BE, EA, EC, the
rectangle GMH is equal to the fquare of ML. Add the com-
mon fquare of NH, (having bifected GH in N), and the fquare
of NM is equal to the fquares of the given ftraight lines NH
and ML, or K; therefore the fquare of NM, and its fide
NM, is given, as alfo the point M, viz. by taking the ftraight
line NM, the fquare of which is equal to the fquares of NH,
ML. Draw ML equal to K, at right angles to GM; and be-
caufe ML is given in pofition and magnitude, therefore the
point L is given; join LG, LH; then the triangle LGH is
that which was to be found, for the fquare of NM is equal to
the fquares of NH and ML, and taking away the common

s 6. 2.

fquare of NH, the rect-
angle GMH is equal to
the fquare of ML; there-
fore as GM to ML, fo is
ML to MH, and the tri-
angle LGM is therefore
equiangular to HLM, and
the angle HLM equal to
the angle LGM, and the

h 6. 6.

ftraight line LM, drawn from the vertex of the triangle making
the angle HLM equal to LGH, is perpendicular to the bafe and
equal to the given ftraight line K, as was required; and the
ratio of the fides GL, LH is the fame with the ratio of GM to
ML, that is, with the ratio of the ftraight line which is made
up of GN the half of the given bafe and of NM, the fquare of
which is equal to the fquares of GN and K, to the ftraight
line K.

And whether this ratio of GM to ML is greater or lefs
than the ratio of the fides of any other triangle upon the bafe
GH, and of which the altitude is equal to the ftraight line K,
that
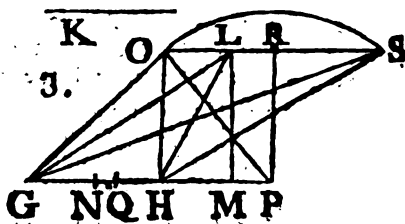
that is, the vertex of which is in the parallel to GH drawn through the point L, may be thus found. Let OGH be any such triangle, and draw OP, making the angle HOP equal to the angle OGH; therefore, as before, GP, PO, PH are proportionals, and PO cannot be equal to LM, becaufe the rectangle GPH would be equal to the rectangle GMH, which is impoffible; for the point P cannot fall upon M, becaufe O would then fall on L; nor can PO be lefs than LM, therefore it is greater; and confequently the rectangle GPH is greater than the rectangle GMH, and the ftraight line GP greater than GM: Therefore the ratio of GM to MH is greater than the ratio of GP to PH, and the ratio of the fquare of GM to the fquare of ML is therefore [i] greater than the ratio of the fquare of GP to the fquare of PO, and the ratio of the ftraight line GM to ML greater than the ratio of GP to PO. But as GM to ML, fo is GL to LH; and as GP to PO, fo is GO to OH; therefore the ratio of GL to LH is greater than the ratio of GO to OH; wherefore the ratio of GL to LH is the greateft of all others; and confequently the given ratio of the greater fide to the lefs muft not be greater than this ratio.

But if the ratio of the fides be not the fame with this greateft ratio of GM to ML, it muft neceffarily be lefs than it: Let any lefs ratio be given, and the fame things being fuppofed, viz. that GH is the bafe, and K equal to the altitude of the triangle, it may be found as follows. Divide GH in the point Q, fo that the ratio of GQ to QH may be the fame with the given ratio of the fides; and as GQ to QH, fo make GP to PQ, and fo will [f] PQ be to PH; wherefore the fquare [f] 19. 5. of GP is to the fquare of PQ, as [i] the ftraight line GP to PH: And becaufe GM, ML, MH are proportionals, the fquare of GM is to the fquare of ML, as [i] the ftraight line GM to MH: But the ratio of GQ to QH, that is, the ratio of GP to PQ, is lefs than the ratio of GM to ML; and therefore the ratio of the fquare of GP to the fquare of PQ is lefs than the ratio of the fquare of GM to that of ML; and confequently the ratio of the ftraight line GP to PH is lefs than the ratio of GM to MH; and, by divifion, the ratio of GH to HP is lefs than that of GH to HM; wherefore [k] the ftraight line HP is [k] 10. 5. greater than HM, and the rectangle GPH, that is, the fquare of PQ, greater than the rectangle GMH, that is, than the

fquare

'fquare of ML, and the ftraight line PQ is therefore greater
than ML.   Draw LR parallel to GP, and from P draw PR at
right angles to GP: Becaufe PQ is greater than ML, or PR,
the circle defcribed from the centre P, at the diftance PQ,
muft neceffarily cut LR in two points; let thefe be O, S, and
join OG, OH; SG, SH; each of the triangles OGH, SGH
have the things mentioned to be given in the propofition:
Join OP, SP; and becaufe as GP to PQ, or PO, fo is PO
to PH, the triangle OGP is equiangular to HOP; as, there-
fore, OG to GP, fo is HO to OP, and, by permutation, as
GO to OH, fo is GP to PO, or PQ; and fo is GQ to QH:
Therefore the triangle OGH has the ratio of its fides GO, OH
the fame with the given ratio of GQ to QH; and the perpen-
dicular has to the bafe the given ratio of K to GH, becaufe the
perpendicular is equal to LM, or K: The like may be fhewn in
the fame way of the triangle SGH.

This conftruction by which the triangle OGH is found, is
fhorter than that which would be deduced from the demon-
ftration of the datum, by reafon that the bafe GH is given
in pofition and magnitude, which was not fuppofed in the
demonftration: The fame thing is to be obferved in the next
propofition.

M.   **P R O P. LXXXI.**

IF the fides about an angle of a triangle be unequal
and have a given ratio to one another, and if the
perpendicular from that angle to the bafe divides it into
fegments that have a given ratio to one another, the
triangle is given in fpecies.

Let ABC be a triangle, the fides of which about the angle
BAC are unequal, and have a given ratio to one another, and
let the perpendicular AD to the bafe BC divide it into the feg-
ments BD, DC which have a given ratio to one another, the
triangle ABC is given in fpecies.

Let AB be greater than AC, and make the angle CAE
equal to the angle ABC; and becaufe the angle AEB is com-
mon to the triangles ABE, CAE, they are a equiangular to
one another: Therefore as AB to BE, fo is CA to AE, and,

a 4. 6.

by

by permutation, as AB to AC, so BE to
EA, and so is EA to EC: But the ratio of
BA to AC is given, therefore the ratio
of BE to EA, as also the ratio of EA to
EC, is given; wherefore [b] the ratio of
BE to EC, as also [c] the ratio of EC to
CB is given: And the ratio of BC to CD
is given [d], because the ratio of BD to
DC is given; therefore [b] the ratio of EC
to CD is given, and consequently [d] the
ratio of DE to EC: And the ratio of EC

b 9. dat.

c cor. 6. dat.

d 7. dat.

to EA was shewn to be given, therefore [b] the ratio of DE to EA
is given: And ADE is a right angle, wherefore [e] the triangle
ADE is given in species, and the angle AED given: And the
ratio of CE to EA is given, therefore [f] the triangle AEC is gi-
ven in species, and consequently the angle ACE is given, as
also the adjacent angle ACB. In the same manner, because the
ratio of BE to EA is given, the triangle BEA is given in spe-
cies, and the angle ABE is therefore given: And the angle
ACB is given; wherefore the triangle ABC is given [g] in spe-
cies.
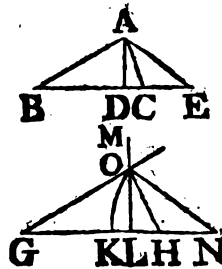
e 41. dat.

f 44. dat.

g 43. dat.

But the ratio of the greater side BA to the other AC must
be less than the ratio of the greater segment BD to DC: Be-
cause the square of BA is to the square of AC, as the squares
of BD, DA to the squares of DC, DA; and the squares of
BD, DA have to the squares of DC, DA a less ratio than the
square of BD has to the square of DC †, because the square of
BD is greater than the square of DC; therefore the square of
BA has to the square of AC a less ratio than the square of BD
has to that of DC: And consequently the ratio of BA to AC
is less than the ratio of BD to DC.

This being premised, a triangle which shall have the things
mentioned to be given in the proposition, and to which the
triangle ABC is similar, may be found thus: Take a straight
line GH given in position and magnitude, and divide it in K,
so that the ratio of GK to KH may be the same with the given
ratio of BA to AC: Divide also GH in L, so that the ratio
of

---

† If A be greater than B, and C any
third magnitude; then A and C toge-
ther have to B and C together a less
ratio than A has to B.
Let A be to B as C to D, and be-
cause A is greater than B, C is greater

than D: But as A is to B, so A and C
to B and D; and A and C have to B
and C a less ratio than A and C have
to B and D, because C is greater than
D, therefore A and C have to B and
C a less ratio than A to B.

of GL to LH may be the fame with the given ratio to of BD to DC, and draw LM at right angles to GH: And becaufe the ratio of the fides of a triangle is lefs than the ratio of the fegments of the bafe, as has been fhewn, the ratio of GK to KH is lefs than the ratio of GL to LH; wherefore the point L muft fall betwixt K and H: Alfo make as GK to KH, fo GN to NK, and fo fhall [h] NK be to NH. And from the centre N, at the diftance NK, defcribe a circle, and let its circumference meet LM in O, and join OG, OH; then OGH 'is the triangle which was to be defcribed: Becaufe GN is to NK, or NO, as NO to NH, the triangle OGN is equiangular to HON; therefore as OG to GN, fo is HO to ON, and, by permutation, as GO to OH, fo is GN to NO, or NK, that is, as GK to KH, that is, in the given ratio of the fides, and, by the conftruction, GL, LH have to one another the given ratio of the fegments of the bafe.

[h] 19. 5.

P R O P. LXXXII.

60.

IF a parallelogram given in fpecies and magnitude be increafed or diminifhed by a gnomon given in magnitude, the fides of the gnomon are given in magnitude.

Firft, let the parallelogram AB given in fpecies and magnitude be increafed by the given gnomon ECBDFG, each of the ftraight lines CE, DF is given.

Becaufe AB is given in fpecies and magnitude, and that the gnomon ECBDFG is given, therefore the whole fpace AG is given in magnitude: But AG is alfo given in fpecies, becaufe it is fimilar [a] to AB; therefore the fides of AG are given [b]: Each of the ftraight lines AE, AF is therefore given; and each of the ftraight lines CA, AD is given [b], therefore each of the remainders EC, DF is given [c].

[a] { 2. def.
    { 2. and
    { 24. 6.
[b] 60. dat.
[c] 4. dat.

Next, let the parallelogram AG, given in fpecies and magnitude, be diminifhed by the given gnomon ECBDFG, each of the ftraight lines CE, DF is given.

Becaufe the parallelogram AG is given, as alfo its gnomon ECBDFG, the remaining fpace AB is given in magnitude:

magnitude : But it is alſo given in ſpecies ; becauſe it is ſimilar [a]
to AG ; therefore [b] its ſides CA, AD are given, and each of the
ſtraight lines EA, AF is given ; therefore EC, DF are each of
them given.

    The gnomon and its ſides CE, DF may be found thus in
the firſt caſe. Let H be the given ſpace to which the gnomon
muſt be made equal, and find [d] a parallelogram ſimilar to AB
and equal to the figures AB and H together, and place its
ſides AE, AF from the point A, upon the ſtraight lines AC,
AD, and complete the parallelogram AG, which is about the
ſame diameter [e] with AB ; becauſe therefore AG is equal to
both AB and H, take away the common part AB, the remain-
ing gnomon ECBDFG is equal to the remaining figure H ;
therefore a gnomon equal to H, and its ſides CE, DF are found :
And in like manner they may be found in the other caſe, in
which the given figure H muſt be leſs than the figure FE from
which it is to be taken.

<div align="right">

[a] 2. def.
2. and
14. 5.
[b] 60. dat.

</div>

<div align="right">

[d] 25. 6.

</div>

<div align="right">

[e] 26. 6.

</div>

PROP. LXXXIII.

<div align="right">52.</div>

IF a parallelogram equal to a given ſpace be applied to
    a given ſtraight line, deficient by a parallelogram given
in ſpecies, the ſides of the defect are given.

    Let the parallelogram AC equal to a given ſpace be applied
to the given ſtraight line AB, deficient by the parallelogram
BDCL given in ſpecies, each of the ſtraight lines CD, DB are
given.

    Biſect AB in E ; therefore EB is given in magnitude, up-
on EB deſcribe [a] the parallelogram EF ſimilar to DL and ſimi-
larly placed ; therefore EF is given in
ſpecies, and is about the ſame diameter [b]
with DL ; let BCG be the diameter, and
conſtruct the figure ; therefore, becauſe
the figure EF given in ſpecies is deſcri-
bed upon the given ſtraight line EB,
EF is given [c] in magnitude, and the
gnomon ELH is equal [d] to the given
figure AC ; therefore [e] ſince EF is diminiſhed by the given
gnomon ELH, the ſides EK, FH of the gnomon are given ; but
EK is equal to DC, and FH to DB ; wherefore CD, DB are
each of them given.

<div align="right">

[a] 18. 6.

[b] 26. 6.

[c] 56. dat.
[d] 36. and
43. 1.
[e] 82. dat.

</div>

<div align="right">This</div>

This demonstration is the analysis of the problem in the
28th prop. of book 6. the construction and demonstration of
which proposition is the composition of the analysis; and be-
cause the given space AC or its equal the gnomon ELH is to
be taken from the figure EF described upon the half of AB si-
milar to BC, therefore AC must not be greater than EF, as is
shewn in the 27th prop. b. 6.

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P R O P.   LXXXIV.

</div>

59.

IF a parallelogram equal to a given space be applied to
a given straight line, exceeding by a parallelogram
given in species; the sides of the excess are given.

    Let the parallelogram AC equal to a given space be applied
to the given straight line AB, exceeding by the parallelogram
BDCL given in species; each of the straight lines CD, DB are
given.
    Bisect AB in E; therefore EB is given in magnitude: Upon

a 18. 6.
EB describe *a* the parallelogram EF similar to LD, and similar-
ly placed; therefore EF is given in species, and is about the

b 26. 6.
same diameter *b* with LD. Let CBG be
the diameter, and construct the figure:
Therefore, because the figure EF given
in species is described upon the given
straight line EB, EF is given in magni-

c 56. dat.
tude *c*, and the gnomon ELH is equal

d 36. and
43. 1.
to the given figure *d* AC ; wherefore,
since EF is increased by the given gnomon ELH, its sides EK,

e 82. dat.
FH are given *e* ; but EK is equal to CD, and FH to BD ; there-
fore CD, DB are each of them given.
    This demonstration is the analysis of the problem in the 29th
prop. book 6. the construction and demonstration of which is
the composition of the analysis.
    COR. If a parallelogram given in species be applied to a gi-
ven straight line, exceeding by a parallelogram equal to a given
space ; the sides of the parallelogram are given.
    Let the parallelogram ADCE given in species be applied to
the given straight line AB exceeding by the parallelogram
BDCG equal to a given space ; the sides AD, DC of the paral-
lelogram are given.

<div align="right">

Draw

</div>

Draw the diameter DE of the parallelogram AC, and con-
ſtruct the figure. Becauſe the parallelogram AK is equal [a] to a 43. 1.
BC which is given, therefore AK is
given; and BK is ſimilar [b] to AC, there-
fore BK is given in ſpecies. And ſince b 24. 6.
the parallelogram AK given in magni-
tude is applied to the given ſtraight line
AB, exceeding by the parallelogram BK
given in ſpecies, therefore, by this pro-
poſition, BD, DK the ſides of the exceſs are given, and the
ſtraight line AB is given; therefore the whole AD, as alſo DC,
to which it has a given ratio, is given.
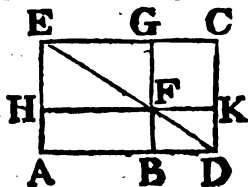
P R O B.

To apply a parallelogram ſimilar to a given one to a given
ſtraight line AB, exceeding by a parallelogram equal to a given
ſpace.

To the given ſtraight line AB apply [c] the parallelogram AK c 29. 6.
equal to the given ſpace, exceeding by the parallelogram BK ſi-
milar to the one given. Draw DF the diameter of BK, and
through the point A draw AE parallel to BF meeting DF pro-
duced in E, and complete the parallelogram AC.

The parallelogram BC is equal [a] to AK, that is, to the given
ſpace; and the parallelogram AC is ſimilar [b] to BK; therefore
the parallelogram AC is applied to the ſtraight line AB ſimilar
to the one given and exceeding by the parallelogram BC which
is equal to the given ſpace.

P R O P. LXXXV. 84.

IF two ſtraight lines contain a parallelogram given in
magnitude, in a given angle; if the difference of the
ſtraight lines be given, they ſhall each of them be given.

Let AB, BC contain the parallelogram AC given in magni-
tude, in the given angle ABC, and let the exceſs of BC above
AB be given; each of the ſtraight lines AB, BC is given.

Let DC be the given exceſs of BC above
BA, therefore the remainder BD is equal
to BA. Complete the parallelogram AD;
and becauſe AB is equal to BD, the ratio
of AB to BD is given; and the angle ABD
is given, therefore the parallelogram AD is
given in ſpecies; and becauſe the given parallelogram AC is
applied to the given ſtraight line DC, exceeding by the paral- a 84. 4at.
lelogram AD given in ſpecies, the ſides of the exceſs are given [a];
therefore

therefore BD is given ; and DC is given, wherefore the whole BC is given : And AB is given, therefore AB, BC are each of them given.

85.

PROP. LXXXVI.

IF two ſtraight lines contain a parallelogram given in magnitude, in a given angle ; if both of them together be given, they ſhall each of them be given.

Let the two ſtraight lines AB, BC contain the parallelogram AC given in magnitude, in the given angle ABC, and let AB, BC together be given ; each of the ſtraight lines AB, BC is given.

Produce CB, and make BD equal to AB, and complete the parallelogram ABDE. Becauſe DB is equal to BA, and the angle ABD given, becauſe the adjacent angle ABC is given, the parallelogram AD is given in ſpecies : And becauſe AB, BC together are given, and AB is equal to BD ; therefore DC is given : And becauſe the given parallelogram AC is applied to the given ſtraight line DC, deficient by the parallelogram AD given in

a 83. dat. ſpecies, the ſides AB, BD of the defect are given [a] ; and DC is given, wherefore the remainder BC is given ; and each of the ſtraight lines AB, BC is therefore given.
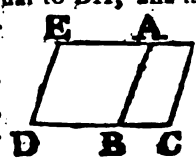
87.

PROP. LXXXVII.

IF two ſtraight lines contain a parallelogram given in magnitude, in a given angle ; if the exceſs of the ſquare of the greater above the ſquare of the leſſer be given, each of the ſtraight lines ſhall be given.

Let the two ſtraight lines AB, BC contain the given parallelogram AC in the given angle ABC ; if the exceſs of the ſquare of BC above the ſquare of BA be given ; AB and BC are each of them given.

Let the given exceſs of the ſquare of BC above the ſquare of BA be the rectangle CB, BD ; take this from the ſquare

a. 2. 2. of BC, the remainder, which is [a] the rectangle BC, CD is equal to the ſquare of AB ; and becauſe the angle ABC and the parallelogram AC is given, the ratio of the rectangle

b 62. dat. of the ſides AB, BC to the parallelogram AC is given [b] ; and AC is given, therefore the rectangle AB, BC is given ; and the rectangle CB, BD is given ; therefore the ratio of the rectangle
angle

angle CB, BD to the rectangle AB, BC, that is [d], the ratio of the [e 1. 6.]
straight line DB to BA is given; therefore [d] the ratio of the [d 54. dat.]
square of DB to the square of BA is gi-
ven: And the square of BA is equal to
the rectangle BC, CD: wherefore the ra-
tio of the rectangle BC, CD to the square
of BD is given, as also the ratio of four
times the rectangle BC, CD to the square
of BD; and, by composition [e], the ratio of four times the rect- [e 7. dat.]
angle BC, CD together with the square of BC to the square
of BD is given: But four times the rectangle BC, CD toge-
ther with the square of BD is equal [f] to the square of the straight [f 8. 2.]
lines BC, CD taken together; therefore the ratio of the square
of BC, CD together to the square of BD is given; wherefore
[g] the ratio of the straight line BC together with CD to BD is [g 58. dat.]
given: And, by composition, the ratio of BC together with
CD and DB, that is, the ratio of twice BC to BD, is given;
therefore the ratio of BC to BD is given, as also [c] the ratio of
the square of BC to the rectangle CB, BD: But the rectangle
CB, BD is given, being the given excess of the squares of BC,
BA; therefore the square of BC, and the straight line BC is
given: And the ratio of BC to BD, as also of BD to BA has
been shown to be given; therefore [h] the ratio of BC to BA is [h 9. dat.]
given; and BC is given, wherefore BA is given.
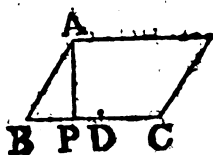
The preceding demonstration is the analysis of this problem, viz.

A parallelogram AC which has a given angle ABC being gi-
ven in magnitude, and the excess of the square of BC one of
its sides above the square of the other BA being given; to find
the sides: And the composition is as follows.

Let EFG be the given angle to which the angle ABC is re-
quired to be equal, and from any point E in FE, draw EG
perpendicular to FG; let the rect-
angle EG, GH be the given space
to which the parallelogram AC is
to be made equal; and the rectangle
HG, GL, be the given excess of the
squares of BC, BA.

Take, in the straight line GE,
GK equal to FE, and make GM double of GK; join ML,
and in GL produced, take LN equal to LM: Bisect GN in O,
and between GH, GO find a mean proportional BC: As OG
to GL, so make CB to BD; and make the angle CBA equal
to

to CFE, and as LG to GK fo make DB to BA; and complete
the parallelogram AC: AC is equal to the rectangle EG, GH,
and the excefs of the fquares of CB, BA is equal to the rect-
angle HG, GL.

Becaufe as CB to BD, fo is OG to GL, the fquare of CB
a 1. 6.   is to the rectangle CB, BD as [a] the rectangle HG, GO to
the rectangle HG, GL: And the fquare of CB is equal to the
rectangle HG, GO, becaufe GO, BC, GH are proportionals;
b 14. 5.   therefore the rectangle CB, BD is equal [b] to HG, GL. And
becaufe as CB to BD, fo is OG to GL; twice CB is to BD,
as twice OG, that is, GN, to GL; and, by divifion, as BC
together with CD is to BD, fo is NL, that is, LM, to LG:
c 22. 6.   Therefore [c] the fquare of BC together with CD is to the fquare
of BD, as the fquare of ML to the fquare of LG: But the
d 8. 2.   fquare of BC and CD together is equal [d] to four times the
rectangle BC, CD together with the fquare of BD; therefore
four times the rectangle BC, CD together with the fquare of
BD is to the fquare of BD, as the fquare of ML to the fquare
of LG: And, by divifion, four times the rectangle BC, CD is
to the fquare of BD, as the fquare of MG to the fquare of
GL; wherefore the rectangle BC, CD is to the fquare of BD,
as (the fquare of KG the half of MG to the fquare of GL,
that is, as) the fquare of AB to the fquare of BD, becaufe as
LG to GK, fo DB was made to BA: Therefore [b] the rectan-
gle BC, CD is equal to the fquare of AB. To each of thefe add
the rectangle CB, BD, and the fquare of BC becomes equal
to the fquare of AB together with the rectangle CB, BD;
therefore this rectangle, that is, the given rectangle HG, GL
is the excefs of the fquares of BC, AB. From the point A,
draw AP perpendicular to BC, and becaufe the angle ABP
is equal to the angle EFG, the triangle ABP is equiangular
to EFG: And DB was made to BA, as LG to GK; therefore
as the rectangle CB, BD to CB, BA, fo is the rectangle HG,
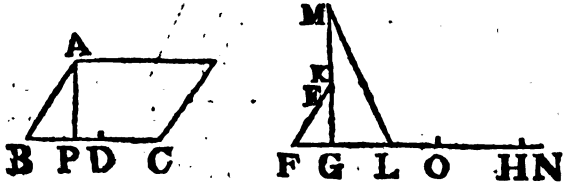
GL to HG, GK; and as the rectangle CB, BA to AP, BC,
fo is (the ftraight line BA to AP, and fo is FE or GK to
EG,

EG, and fo is) the rectangle HG, GK to HG, GE; therefore,
ex aequali, as the rectangle CB, BD to AP, BC, fo is the rect-
angle HG, GL to EG, GH: And the rectangle CB, BD is
equal to HG, GL; therefore the rectangle AP, BC, that is,
the parallelogram AC, is equal to the given rectangle EG, GH.

PROP. LXXXVIII.

N.

IF two ftraight lines contain a parallelogram given in
magnitude, in a given angle; if the fum of the fquares
of its fides be given, the fides fhall each of them be given.

Let the two ftraight lines AB, BC contain the parallelogram
ABCD given in magnitude in the given angle ABC, and let the
fum of the fquares of AB, BC be given; AB, BC are each of
them given.
Firft, let ABC be a right angle; and becaufe twice the rect-
angle contained by two equal ftraight lines is equal to both
their fquares; but if two ftraight lines are un-
equal, twice the rectangle contained by them is
lefs than the fum of their fquares, as is evident
from the 7th prop. b. 2. Elem. therefore twice
the given fpace, to which fpace the rectangle of which the fides
are to be found is equal, muft not be greater than the given
fum of the fquares of the fides: And if twice that fpace be e-
qual to the given fum of the fquares, the fides of the rectangle
muft neceffarily be equal to one another: Therefore in this
cafe defcribe a fquare ABCD equal to the given rectangle, and
its fides AB, BC are thofe which were to be found: For the
rectangle AC is equal to the given fpace, and the fum of the
fquares of its fides AB, BC is equal to twice the rectangle AC,
that is, by the hypothefis, to the given fpace to which the fum
of the fquares was required to be equal.
But if twice the given rectangle be not equal to the given
fum of the fquares of the fides, it muft be lefs than it, as
has been fhown. Let ABCD be the rectangle, join AC and
draw BE perpendicular to it, and complete the rectangle
AEBF, and defcribe the circle ABC about the triangle ABC;
AC is its diameter [a]: And becaufe the triangle ABC is fimi- a Cor. 5. 4.
lar [b] to AEB, as AC to CB, fo is AB to BE; therefore the b 8. 6.
rectangle AC, BE is equal to AB, BC; and the rectangle AB,

BC

BC is given, wherefore AC, BE is given : And because the sum of the squares of AB, BC is given, the square of AC which is

c 47. 1.

equal ^c to that sum is given ; and AC itself is therefore given in magnitude : Let AC be likewise given in position, and the

d 32. dat.

point A ; therefore AF is given ^d in position : And the rectangle AC, BE is given, as has been shewn, and AC is

e 61. dat.

given, wherefore ^e BE is given in magnitude, as also AF which is equal to it ; and AF is also given in position, and

f 30. dat.

the point A is given ; wherefore ^f the point F is given, and the straight line

g 31. dat.

FB in position ^g : And the circumference

h 28. dat.

ABC is given in position, wherefore ^h the point B is given: And the points A, C are given; therefore the straight lines AB,

i 29. dat.

BC are given ⁱ in position and magnitude.
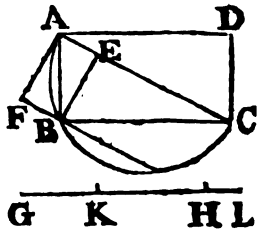
The sides AB, BC of the rectangle may be found thus: Let the rectangle GH, GK be the given space to which the rectangle AB, BC is equal ; and let GH, GL be the given rectangle to which the sum of the squares of AB, BC is equal:

k 14. 2.

Find ^k a square equal to the rectangle GH, GL: And let its side AC be given in position; upon AC as a diameter describe the semicircle ABC, and as AC to GH, so make GK to AF, and from the point A place AF at right angles to AC : There-

l 16. 6.

fore the rectangle CA, AF is equal ^l to GH, GK ; and, by the hypothesis, twice the rectangle GH, GK is less than GH, GL, that is, than the square of AC ; wherefore twice the rectangle CA, AF is less than the square of AC, and the rectangle CA, AF itself less than half the square of AC, that is, than the rectangle contained by the diameter AC and its half; wherefore AF is less than the semidiameter of the circle, and consequently the straight line drawn through the point F parallel to AC must meet the circumference in two points: Let B be either of them, and join AB, BC, and complete the rectangle ABCD ; ABCD is the rectangle which was to be found : Draw

m 34. 1.

BE perpendicular to AC ; therefore BE is equal ^m to AF, and because the angle ABC in a semicircle is a right angle, the rect-

b 8. 6.

angle AB, BC is equal ^b to AC, BE, that is, to the rectangle CA, AF, which is equal to the given rectangle GH, GK : And the squares of AB, BC are together equal ^c to the square of AC, that is, to the given rectangle GH, GL.

But

But if the given angle ABC of the parallelogram AC be not a right angle, in this cafe, becaufe ABC is a given angle, the ratio of the rectangle contained by the fides AB, BC to the parallelogram AC is given [n]; and AC is given, therefore the rect- **a** 62. dat. angle AB, BC is given; and the fum of the fquares of AB, BC is given; therefore the fides AB, BC are given by the preceding cafe.

The fides AB, BC and the parallelogram AC may be found thus: Let EFG be the given angle of the parallelogram, and from any point E in FE draw EG perpendicular to FG; and let the rectangle EG, FH be the given fpace to which the parallelogram is to be made equal, and let EF, FK be the given rectangle to which the fum of the fquares of the fides is to be equal. And, by the preceding cafe, find the fides of a rectangle which is equal to the given rectangle EF, FH, and the fquares of the fides of which are together equal to the given rectangle EF, FK; therefore, as was fhewn in that cafe, twice the rectangle EF, FH muft not be greater than the rectangle EF, FK; let it be fo, and let AB, BC be the fides of the rectangle joined in the angle ABC equal to the given angle EFG; and complete the parallelogram ABCD, which will be that which was to be found: Draw AL perpendicular to BC, and becaufe the angle ABL is equal to EFG, the triangle ABL is equiangular to EFG; and the parallelogram AC, that is, the rectangle AL, BC is to the rectangle AB, BC as (the ftraight line AL to AB, that is, as EG to EF, that is, as) the rectangle EG, FH to EF, FH; and, by the conftruction, the rectangle AB, BC is equal to EF, FH, therefore the rectangle AL, BC, or, its equal, the parallelogram AC, is equal to the given rectangle EG, FH; and the fquares of AB, BC are together equal, by conftruction, to the given rectangle EF, FK.
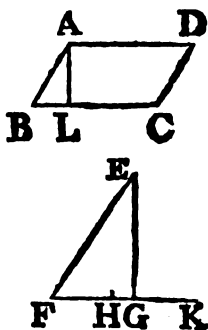
F f

PROP.

P R O P. LXXXIX.

IF two ſtraight lines contain a given parallelogram in a given angle, and if the exceſs of the ſquare of one of them above a given ſpace, has a given ratio to the ſquare of the other; each of the ſtraight lines ſhall be given.

Let the two ſtraight lines AB, BC contain the given parallelogram AC in the given angle ABC, and let the exceſs of the ſquare of BC above a given ſpace have a given ratio to the ſquare of AB, each of the ſtraight lines AB, BC is given.

Becauſe the exceſs of the ſquare of BC above a given ſpace has a given ratio to the ſquare of BA, let the rectangle CB, BD be the given ſpace; take this from the ſquare of BC, the

a 2. 2.   remainder, to wit, the rectangle ᵃ BC, CD has a given ratio to the ſquare of BA: Draw AE perpendicular to BC, and let the ſquare of BF be equal to the rectangle BC, CD, then be-cauſe the angle ABC, as alſo BEA, is given,

b 43. dat.   the triangle ABE is given ᵇ in ſpecies, and the ratio of AE to AB given: And becauſe the ratio of the rectangle BC, CD, that is, of the ſquare of BF to the ſquare of BA, is given, the ratio of the ſtraight line BF to BA

c 58. dat.   is given ᶜ; and the ratio of AE to AB is given, wherefore ᵈ the
d 9. dat.   ratio of AE to BF is given; as alſo the ratio of the rectangle
e 35. 1.   AE, BC, that is, ᵉ of the parallelogram AC to the rectangle FB, BC; and AC is given, wherefore the rectangle FB, BC is given. The exceſs of the ſquare of BC above the ſquare of BF, that is, above the rectangle BC, CD, is given, for it is equal ᵃ to the given rectangle CB, BD; therefore, becauſe the rectangle contained by the ſtraight lines FB, BC is given, and alſo the exceſs of the ſquare of BC above the ſquare of BF; FB, BC

f 87. dat.   are each of them given ᶠ; and the ratio of FB to BA is given; therefore, AB, BC are given.
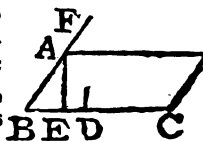
The Compoſition is as follows.

Let GHK be the given angle to which the angle of the pa-rallelogram is to be made equal, and from any point G in HG, draw GK perpendicular to HK; let GK, HL be the rect-angle

angle to which the parallelogram is to be
made equal, and let LH, HM be the rect-
angle equal to the given space which is to
be taken from the square of one of the sides;
and let the ratio of the remainder to the
square of the other side be the same with the
ratio of the square of the given straight line NH to the square
of the given straight line HG.
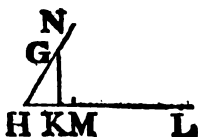
By help of the 87th dat. find two straight lines BC, BF
which contain a rectangle equal to the given rectangle NH,
HL, and such that the excess of the square
of BC above the square of BF be equal to
the given rectangle LH, HM; and join CB,
BF in the angle FBC equal to the given
angle GHK: And as NH to HG, so make
FB to BA, and complete the parallelogram
AC, and draw AE perpendicular to BC; then AC is equal to
the rectangle GK, HL; and if from the square of BC, the gi-
ven rectangle LH, HM be taken, the remainder shall have to
the square of BA the same ratio which the square of NH has to
the square of HG.
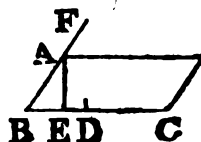
Because, by the construction, the square of BC is equal to
the square of BF together with the rectangle LH, HM; if
from the square of BC there be taken the rectangle LH, HM,
there remains the square of BF which has g to the square of   g 22. 6.
BA the same ratio which the square of NH has to the square
of HG, because, as NH to HG, so FB was made to BA; but
as HG to GK, so is BA to AE, because the triangle GHK is
equiangular to ABE; therefore, ex aequali, as NH to GK so is
FB to AE; wherefore h the rectangle NH, HL is to the rect-   h 1. 6.
angle GK, HL, as the rectangle FB, BC to AE, BC; but by
the construction, the rectangle NH, HL is equal to FB, BC;
therefore k the rectangle GK, HL is equal to the rectangle AE,   k 14. 5.
BC, that is, to the parallelogram AC.

The analysis of this problem might have been made as in the
86th prop. in the Greek, and the composition of it may be made
as that which is in prop. 87th of this edition.

P R O P.

O.

PROP. XC.

IF two ftraight lines contain a given parallelogram in a given angle, and if the fquare of one of them together with the fpace which has a given ratio to the fquare of the other be given, each of the ftraight lines fhall be given.

Let the two ftraight lines AB, BC contain the given parallelogram AC in the given angle ABC, and let the fquare of BC together with the fpace which has a given ratio to the fquare of AB be given, AB, BC are each of them given.

Let the fquare of BD be the fpace which has the given ratio to the fquare of AB ; therefore, by the hypothefis, the fquare of BC together with the fquare of BD is given. From the point A, draw AE perpendicular to BC; and becaufe the angles

a 43. dat.

ABE, BEA are given, the triangle ABE is given a in fpecies ; therefore the ratio of BA to AE is given : And becaufe the ratio of the fquare of BD to the fquare of BA is given, the ra-

b 58. dat.

tio of the ftraight line BD to BA is given b ; and the ratio of

c 9. dat.

BA to AE is given ; therefore c the ratio of AE to BD is given, as alfo the ratio of the rectangle AE, BC, that is, of the parallelogram AC to the rectangle DB, BC ; and AC is given, therefore the rectangle DB, BC is given ; and the fquare of

d 88. dat.
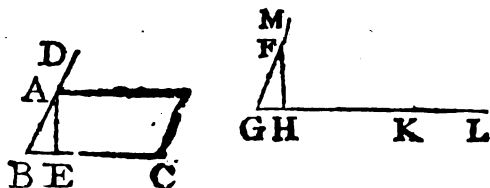
BC together with the fquare of BD is given ; therefore d becaufe the rectangle contained by the two ftraight lines DB, BC is given, and the fum of their fquares is given : The ftraight lines DB, BC are each of them given ; and the ratio of DB to BA is given ; therefore AB, BC are given.

The compofition is as follows.

Let FGH be the given angle to which the angle of the parallelogram is to be made equal, and from any point F in GF draw FH perpendicular to GH ; and let the rectangle FH, GK be that to which the parallelogram is to be made equal ; and let the rectangle KG, GL be the fpace to which the fquare

of

of one of the sides of the parallelogram together with the space which has a given ratio to the square of the other side, is to be made equal ; and let this given ratio be the same which the square of the given straight line MG has to the square of GF.

By the 88th dat. find two straight lines DB, BC which contain a rectangle equal to the given rectangle MG, GK, and such that the sum of their squares is equal to the given rectangle KG, GL ; therefore, by the determination of the problem in that proposition, twice the rectangle MG, GK must not be greater than the rectangle KG, GL. Let it be so, and join the straight lines DB, BC in the angle DBC equal to the given angle FGH; and, as MG to GF, so make DB to BA, and complete the parallelogram AC: AC is equal to the rect-

angle FH, GK ; and the square of BC together with the square of BD, which, by the construction, has to the square of BA the given ratio which the square of MG has to the square of GF, is equal, by the construction, to the given rectangle KG, GL. Draw AE perpendicular to BC.
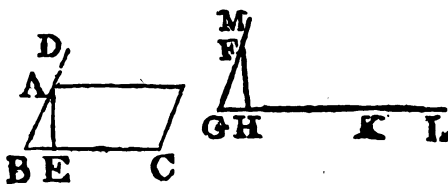
Because, as DB to BA, so is MG to GF ; and as BA to AE, so GF to FH ; ex aequali, as DB to AE, so is MG to FH ; therefore, as the rectangle DB, BC to AE, BC, so is the rectangle MG, GK to FH, GK ; and the rectangle DB, BC is equal to the rectangle MG, GK ; therefore the rectangle AE, BC, that is, the parallelogram AC, is equal to the rectangle FH, GK.

P R O P. XCI.   83.

IF a straight line drawn within a circle given in magnitude cuts off a segment which contains a given angle ; the straight line is given in magnitude.

In the circle ABC given in magnitude, let the straight line AC be drawn, cutting off the segment AEC which contains the given angle AEC ; the straight line AC is given in magnitude. Take D the centre of the circle[a], join AD and produce it a 1. 3.

to E, and join EC : The angle ACE being
a right ^b angle is given ; and the angle

AEC is given ; therefore ^c the triangle
ACE is given in fpecies, and the ratio of
EA to AC is therefore given ; and EA is
given ^d in magnitude, becaufe the circle is

given in magnitude ; AC is therefore gi-

ven ^e in magnitude.
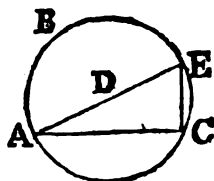
PROP. XCII.

IF a ftraight line given in magnitude be drawn within a
circle given in magnitude, it fhall cut off a fegment
containing a given angle,

Let the ftraight line AC given in magnitude be drawn with-
in the circle ABC given in magnitude ; it fhall cut off a fegment
containing a given angle.

Take D the centre of the circle, join
AD and produce it to E, and join EC :
And becaufe each of the ftraight lines EA,

and AC is given, their ratio is given ^a ; and
the angle ACE is a right angle, therefore

the triangle ACE is given ^b in fpecies,
and confequently, the angle AEC is given.
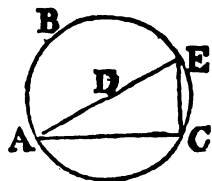
PROP. XCIII.

IF from any point in the circumference of a circle gi-
ven in pofition two ftraight lines be drawn meeting
the circumference and containing a given angle ; if the
point in which one of them meets the circumference again
be given, the point in which the other meets it is alfo gi-
ven.

From any point A in the circumference of a circle ABC given
in pofition, let AB, AC be drawn to the circumference making
the given angle BAC ; if the point B be
given, the point C is alfo given.
Take D the centre of the circle, and
join BD, DC ; and becaufe each of the

points B, D is given, BD is given ^a in po-
fition ; and becaufe the angle BAC is gi-

ven, the angle BDC is given ^b, therefore

becaufe
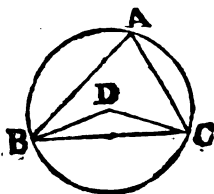
becaufe the ftraight line DC is drawn to the given point D in the ftraight line BD given in pofition in the given angle BDC, DC is given ᶜ in pofition : And the circumference ABC is gi-ᶜ 32. dat. ven in pofition, therefore ᵈ the point C is given. ᵈ 28. dat.

PROP. XCIV.

91.

IF from a given point a ftraight line be drawn touching a circle given in pofition ; the ftraight line is given in pofition and magnitude.

Let the ftraight line AB be drawn from the given point A touching the circle BC given in pofition ; AB is given in pofition and magnitude.

Take D the centre of the circle, and join DA, DB : Becaufe each of the points D, A is given, the ftraight line AD is given ᵃ in pofition and magnitude : And DBA is a right ᵇ angle, wherefore DA is a diameter ᶜ of the circle DBA, defcribed about the triangle DBA ; and that circle is therefore given ᵈ in pofition : And the circle BC is given in pofition, therefore the point B is given ᵉ : The point A is alfo given ; therefore the ftraight line AB is given ᵃ in pofition and magnitude.
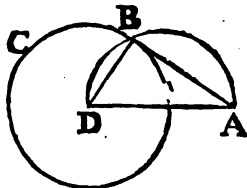
ᵃ 29. dat.
ᵇ 18. 3.
ᶜ Cor. 5. 4.

ᵈ 6. def.

ᵉ 28. dat.

PROP. XCV.

92.

IF a ftraight line be drawn from a given point without a circle given in pofition ; the rectangle contained by the fegments betwixt the point and the circumference of the circle is given.

Let the ftraight line ABC be drawn from the given point A without the circle BCD given in pofition, cutting it in B, C ; the rectangle BA, AC is given.

From the point A, draw ᵃ AD touching the circle ; therefore AD is given ᵇ in pofition and magnitude : And becaufe AD is given, the fquare of AD is given ᶜ which is equal ᵈ to the rectangle BA, AC : Therefore the rectangle BA, AC is given.
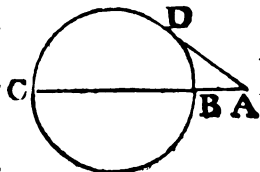
ᵃ 17. 3.

ᵇ 94. dat.

ᶜ 56. dat.
ᵈ 36. 3.

F f 4

PROP.

P R O P. XCVI.

IF a ftraight line be drawn through a given point with-
in a circle given in pofition, the rectangle contained by
the fegments betwixt the point and the circumference of
the circle is given.

Let the ftraight line BAC be drawn through the given point
A within the circle BCE given in pofition; the rectangle BA,
AC is given.

Take D the centre of the circle, join
AD and produce it to the points E, F.
Becaufe the points A, D are given, the
ftraight line AD is given ^a in pofition ;
and the circle BEC is given in pofition;
therefore the points E, F are given ^b, and
the point A is given, therefore EA, AF
are each of them given ^a; and the rect-
angle EA, AF is therefore given ; and it is equal ^c to the rect-
angle BA, AC, which confequently is given.
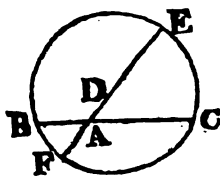
a 29. dat.

b 28. dat.

c 35. 3.

P R O P. XCVII.

94.

IF a ftraight line be drawn within a circle given in
magnitude cutting off a fegment containing a given
angle ; if the angle in the fegment be bifected by a ftraight
line produced till it meets the circumference, the ftraight
lines which contain the given angle fhall both of them to-
gether have a given ratio to the ftraight line which bifects
the angle :   And the rectangle contained by both thefe
lines together which contain the given angle, and the part
of the bifecting line cut off below the bafe of the fegment,
fhall be given.

Let the ftraight line BC be drawn within the circle ABC gi-
ven in magnitude cutting off a fegment containing the given
angle BAC, and let the angle BAC be bifected by the ftraight
line AD; BA together with AC has a given ratio to AD ; and
the rectangle contained by BA and AC together, and the ftraight
line ED cut off from AD below BC the bafe of the fegment, is
given.

Join BD ; and becaufe BC is drawn within the circle ABC
given

given in magnitude cutting off the fegment BAC, containing
the given angle BAC; BC is given ᵃ in magnitude : By the ᵃ 91. dat.
fame reafon BD is given ; therefore ᵇ the ratio of BC to BD ᵇ 1. dat.
is given : And becaufe the angle BAC is bifected by AD, as
BA to AC, fo is ᶜ BE to EC; and, by permutation, as AB ᶜ 3. 6.
to BE, fo is AC to CE; wherefore ᵈ as BA and AC together ᵈ 12. 5.
to BC, fo is AC to CE : And becaufe the angle BAE is equal
to EAC, and the angle ACE to ᵉ ᵉ 21. 3.
ADB, the triangle ACE is equiangu-
lar to the triangle ADB ; therefore
as AC to CE, fo is AD to DB : But
as AC to CE, fo is BA together with
AC to BC; as therefore BA and AC
to BC, fo is AD to DB ; and, by
permutation, as BA and AC to AD,
fo is BC to BD : And the ratio of BC to BD is given, there-
fore the ratio of BA together with AC to AD is given.
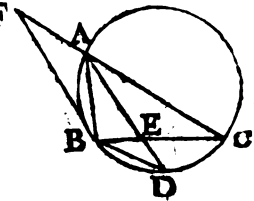
Alfo the rectangle contained by BA and AC together, and
DE is given.

Becaufe the triangle BDE is equiangular to the triangle ACE,
as BD to DE, fo is AC to CE ; and as AC to CE, fo is BA
and AC to BC; therefore as BA and AC to BC, fo is BD to
DE ; wherefore the rectangle contained by BA and AC toge-
ther, and DE, is equal to the rectangle CB, BD : But CB, BD
is given ; therefore the rectangle contained by BA and AC to-
gether, and DE, is given.

Otherwife.

Produce CA, and make AF equal to AB, and join BF ;
and becaufe the angle BAC is double ᵃ of each of the angles ᵃ { 5. &
BFA, BAD, the angle BFA is equal to BAD ; and the angle   { 32. 1.
BCA is equal to BDA, therefore the triangle FCB is equiangu-
lar to ABD : As therefore as FC to CB, fo is AD to DB ; and,
by permutation, as FC, that is, BA and AC together to AD,
fo is CB to BD : And the ratio of CB to BD is given, there-
fore the ratio of BA and AC to AD is given.

And becaufe the angle BFC is equal to the angle DAC,
that is, to the angle DBC, and the angle ACB equal to the
angle ADB; the triangle FCB is equiangular to BDE, as
therefore FC to CB, fo is BD to DE ; therefore the rectangle
contained by FC, that is, BA and AC together, and DE is e-
qual

qual to the rectangle CB, BD, which is given, and therefore
the rectangle contained by BA, AC together, and DE is given.

P.

PROP. XCVIII.

IF a straight line be drawn within a circle given in mag-
nitude, cutting off a segment containing a given angle;
if the angle adjacent to the angle in the segment be bi-
sected by a straight line produced till it meet the circum-
ference again and the base of the segment; the excess of
the straight lines which contain the given angle shall have
a given ratio to the segment of the bisecting line which is
within the circle; and the rectangle contained by the same
excess and the segment of the bisecting line betwixt the
base produced and the point where it again meets the cir-
cumference, shall be given.

Let the straight line BC be drawn within the circle ABC
given in magnitude cutting off a segment containing the given
angle BAC, and let the angle CAF adjacent to BAC be bisect-
ed by the straight line DAE meeting the circumference again
in D, and BC the base of the segment produced in E; the ex-
cess of BA, AC has a given ratio to AD; and the rectangle
which is contained by the same excess and the straight line ED,
is given.

Join BD, and through B, draw BG parallel to DE meeting
AC produced in G: And because BC cuts off from the circle
ABC given in magnitude the seg-
ment BAC containing a given an-

a 91. dat. gle, BC is therefore given ª in mag-
nitude: By the same reason BD is
given, because the angle BAD is e-
qual to the given angle EAF; there-
fore the ratio of BC to BD is given:
And because the angle CAE is equal
to EAF, of which CAE is equal to
the alternate angle AGB, and EAF to the interior and opposite
angle ABG; therefore the angle AGB is equal to ABG, and
the straight line AB equal to AG; so that GC is the excess
of
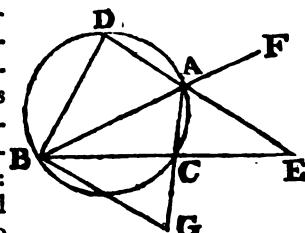
of BA, AC: And becaufe the angle BGC is equal to GAE, that is, to EAF, or the angle BAD; and that the angle BCG is equal to the oppofite interior angle BDA of the quadrilateral BCAD in the circle; therefore the triangle BGC is equiangular to BDA: Therefore as GC to CB, fo is AD to DB; and, by permutation, as GC which is the excefs of BA, AC to AD, fo is CB to BD: And the ratio of CB to BD is given; therefore the ratio of the excefs of BA, AC to AD is given.

And becaufe the angle GBC is equal to the alternate angle DEB, and the angle BCG equal to BDE; the triangle BCG is equiangular to BDE: Therefore as GC to CB, fo is BD to DE; and confequently the rectangle GC, DE is equal to the rectangle CB, BD which is given, becaufe its fides CB, BD are given: Therefore the rectangle contained by the excefs of BA, AC and the ftraight line DE is given.

PROP. XCIX.

95.

IF from a given point in the diameter of a circle given in pofition, or in the diameter produced, a ftraight line be drawn to any point in the circumference, and from that point a ftraight line be drawn at right angles to the firft, and from the point in which this meets the circumference again, a ftraight line be drawn parallel to the firft; the point in which this parallel meets the diameter is given; and the rectangle contained by the two parallels is given.

In BC the diameter of the circle ABC given in pofition, or in BC produced, let the given point D be taken, and from D let a ftraight line DA be drawn to any point, A in the circumference, and let AE be drawn at right angles to DA, and from the point E where it meets the circumference again let EF be drawn parallel to DA meeting BC in F; the point F is given, as alfo the rectangle AD, EF.

Produce EF to the circumference in G, and join AG: Becaufe GEA is a right angle, the ftraight line AG is [a] the diameter of the circle ABC; and BC is alfo a diameter of it; therefore the point H where they meet is the centre of the circle, and confequently H is given: And the point D is given, wherefore DH is given in magnitude: And becaufe AD is parallel

a Cor. 5. 4.

rallel

rallel to FG, and GH equal to HA ; DH is equal ᵇ to HF, and
AD equal to GF : And DH is given, therefore HF is given in
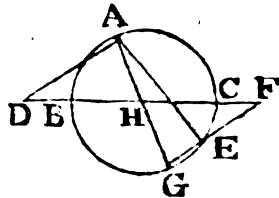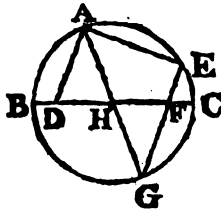
magnitude ; and it is alfo given in pofition, and the point H is
given, therefore ᶜ the point F is given.

And becaufe the ftraight line EFG is drawn from a given
point F without or within the circle ABC given in pofition,
therefore ᵈ the rectangle EF, FG is given : And GF is equal to
AD, wherefore the rectangle AD, EF is given.

Q.

P R O P. C.

IF from a given point in a ftraight line given in pofi-
tion, a ftraight line be drawn to any point in the cir-
cumference of a circle given in pofition ; and from this
point a ftraight line be drawn making with the firft an
angle equal to the difference of a right angle and the angle
contained by the ftraight line given in pofition, and the
ftraight line which joins the given point and the centre of
the circle ; and from the point in which the fecond line
meets the circumference again, a third ftraight line be
drawn making with the fecond an angle equal to that
which the firft makes with the fecond : The point in which
this third line meets the ftraight line given in pofition is
given ; as alfo the rectangle contained by the firft ftraight
line and the fegment of the third betwixt the circumfe-
rence and the ftraight line given in pofition, is given.

Let the ftraight line CD be drawn from the given point C
in the ftraight line AB given in pofition, to the circumference
of the circle DEF given in pofition, of which G is the centre ;
join CG, and from the point D let DF be drawn making the
angle CDF equal to the difference of a right angle and the
angle BCG, and from the point F let FE be drawn making
the

the angle DFE equal to CDF, meeting AB in H : The point H is given ; as also the rectangle CD, FH.

Let CD, FH meet one another in the point K, from which draw KL perpendicular to DF; and let DC meet the circumference again in M, and let FH meet the same in E, and join MG, GF, GH.

Because the angles MDF, DFE are equal to one another, the circumferences MF, DE are equal [a]; and adding or taking away the common part ME, the circumference DM is equal to EF; therefore the straight line DM is equal to the straight line EF, and the angle GMD to the angle [b] GFE; and the angles GMC, GFH are equal to one another, because they are either the same with the angles GMD, GFE, or adjacent to them : And because the angles KDL, LKD are together equal [c] to a right angle, that is, by the hypothesis, to the angles KDL, GCB; the angle GCB or GCH is equal to the angle (LKD, that is, to the angle) LKF or GKH : Therefore the points C, K, H, G are in the circumference of a circle ; and the angle GCK is therefore equal to the angle GHF ; and the angle GMC is equal to GFH, and the straight line GM to GF ; therefore [d] CG is equal to GH, and CM to HF : And because CG is equal to GH, the angle GCH is equal to GHC ; but the angle GCH is given : Therefore GHC is given, and consequently the angle CGH is given; and CG is given in position, and the point G ; therefore [e] GH is given in position ; and CB is also given in position, wherefore the point H is given.
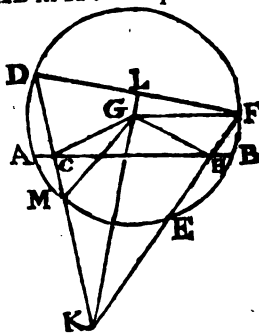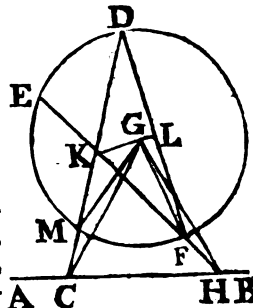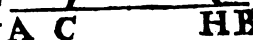
And because HF is equal to CM, the rectangle DC, FH is equal to DC, CM : But DC, CM is given [f], because the point C is given ; therefore the rectangle DC, FH is given.

a 26. 3.

b 8. 1.

c 32. 1.

d 26. 1.

e 32. dat.

f 95. or 96. dat.

F I N I S.

N O T E S

O N

E U C L I D's D A T A.

DEFINITION II.

THIS is made more explicit than in the Greek text, to pre-
vent a miſtake which the author of the ſecond demon-
ſtration of the 24th propoſition in the Greek edition has fallen
into, of thinking that a ratio is given to which another ratio is
ſhown to be equal, though this other be not exhibited in given
magnitudes. See the notes on that propoſition, which is the
13th in this edition. Beſides, by this definition, as it is now gi-
ven, ſome propoſitions are demonſtrated, which in the Greek
are not ſo well done by help of prop. 2.

D E F. IV.

In the Greek text, def. 4. is thus: " Points, lines, ſpaces,
" and angles are ſaid to be given in poſition which have always
" the ſame ſituation;" but this is imperfect and uſeleſs, becauſe
there are innumerable caſes in which things may be given accor-
ding to this definition, and yet their poſition cannot be found;
for inſtance, let the triangle ABC be given in poſition, and let
it be propoſed to draw a ſtraight line BD from the angle at B
to the oppoſite ſide AC which ſhall cut
off the angle DBC which ſhall be the
ſeventh part of the angle ABC; ſuppoſe
this is done, therefore the ſtraight line
BD is invariable in its poſition, that is,
has always the ſame ſituation; for any
other ſtraight line drawn from the point B on either ſide of
BD cuts off an angle greater or leſſer than the ſeventh part of
the angle ABC; therefore, according to this definition, the
ſtraight line BD is given in poſition, as alſo ᵃ the point D in a 28. dat.
which it meets the ſtraight line AC which is given in poſition.
But from the things here given, neither the ſtraight line BD
nor the point D can be found by the help of Euclid's Ele-
ments only, by which every thing in his data is ſuppoſed may
                                                                      be
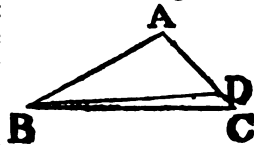
be found. This definition is th erefore of no ufe. We have x·
mended it by adding, " and whi ch are either actually exhibited
" or can be found ;" for nothing is to be reckoned given, which
cannot be found, or is not actually exhibited.

The definition of an angle given by pofition is taken out of
the 4th, and given more diftinctly by itfelf in the definition
marked A.

DEF. XI. XII. XIII. XIV. XV.

The 11th and 12th are omitted, becaufe they cannot be given
in Englifh fo as to have any tolerable fenfe; and, therefore,
wherever the terms defined occur, the words which exprefs
their meaning are made ufe of in their place.

The 13th, 14th, 15th are omitted, as being of no ufe.

It is to be obferved in general of the data in this book, that
they are to be underftood to be given geometrically, not always
arithmetically, that is, they cannot always be exhibited in num-
bers; for inftance, if the fide of a fquare be given, the ratio of
b 44. dat. it to its diameter is given ᵇ geometrically, but not in numbers;
c 2. dat. and the diameter is given ᶜ; but though the number of any e-
qual parts in the fide be given, for example 10. the number of
them in the diameter cannot be given: And the like holds in
many other cafes.

PROPOSITION I.

In this it is fhown that A is to B, as C to D, from this, that
A is to C, as B to D, and then by permutation; but it follows
directly, without thefe two fteps, from 7. 5.

PROP. II.

The limitation added at the end of this propofition between
the inverted commas is quite neceflary, becaufe without it the
propofition cannot always be demonftrated: For the author
having faid * " becaufe A is given, a magnitude equal to it
a 1. def. " can be found ª; let this be C; and becaufe the ratio of A
b 2. def. " to B is given, a ratio which is the fame to it can be found ᵇ,"
adds, " let it be found, and let it be the ratio of C to Δ."
Now, from the fecond definition nothing more follows
than that fome ratio, fuppofe the ratio of E to Z, can be
found, which is the fame with the ratio of A to B; and
when the author fuppofes that the ratio of C to Δ, which is
alfo

* See Dr Gregory's edition of the data.

alſo the ſame with the ratio of **A** to **B**, can be found, he ne-
ceſſarily ſuppoſes that to the three magnitudes **E, Z, C**, a
fourth proportional **Δ** may be found; but this cannot always
be done by the Elements of Euclid; from which it is plain
Euclid muſt have underſtood the Propoſition under the limita-
tion which is now added to his text. An example will make
this clear; let **A** be a given angle,
and **B** another angle to which **A** has
a given ratio, for inſtance, the ratio
of the given ſtraight line **E** to the gi-
ven one **Z**; then, having found an
angle **C** equal to **A**, how can the
angle **Δ** be found to which **C** has the
ſame ratio that **E** has to **Z**? certainly
no way, until it be ſhewn how to find
an angle to which a given angle has
a given ratio, which cannot be done
by Euclid's Elements, nor probably by any Geometry known
in his time. Therefore, in all the propoſitions of this book
which depend upon this ſecond, the above mentioned limitation
muſt be underſtood, though it be not explicitly mentioned.
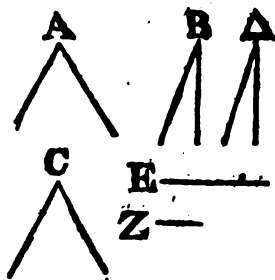
PROP. V.

The order of the Propoſitions in the Greek text between
prop. 4. and prop. 25. is now changed into another which is
more natural, by placing thoſe which are more ſimple before
thoſe which are more complex; and by placing together thoſe
which are of the ſame kind, ſome of which were mixed among
others of a different kind. Thus, prop. 12. in the Greek is now
made the 5th, and thoſe which were the 22d and 23d are made
the 11th and 12th, as they are more ſimple than the propoſitions
concerning magnitudes, the exceſs of one of which above a gi-
ven magnitude has a given ratio to the other, after which theſe
two were placed; and the 24th in the Greek text is, for the
ſame reaſon, made the 13th.

PROP. VI, VII.

Theſe are univerſally true, tho' in the Greek text they are de-
monſtrated by prop. 2. which has a limitation; they are there-
fore now ſhewn without it.

G g   PROP.

P R O P. XII.

In the 23d prop. in the Greek text, which here is the 12th, the words, " μη τας αυτας ἡ," are wrong tranflated by Claud. Hardy, in his edition of Euclid's Data, printed at Paris, ann. 1625, which was the firſt edition of the Greek text; and Dr Gregory follows him in tranſlating them by the words, "etſi non eaſdem," as if the Greek had been ας και μη ἱας αυτας as in prop. 9. of the Greek text. Euclid's meaning is, that the ratios mentioned in the propoſition muſt not be the fame; for, if they were, the propoſition would not be true. Whatever ratio the whole has to the whole, if the ratios of the parts of the firſt to the parts of the other be the fame with this ratio, one part of the firſt may be double, triple, &c. of the other part of it, or have any other ratio to it, and conſequently cannot have a given ratio to it; wherefore, theſe words muſt be rendered by " non autem eaſdem," but not the fame ratios, as Zambertus has tranſlated them in his edition.

P R O P. XIII.

Some very ignorant editor has given a ſecond demonſtration of this propoſition in the Greek text, which has been as ignorantly kept in by Claud. Hardy and Dr Gregory, and has been retained in the tranſlations of Zambertus and others ; Carolus Renaldinus gives it only :   The author of it has thought that a ratio was given if another ratio could be ſhown to be the fame to it, though this laſt ratio be not found : But this is altogether abſurd, becauſe from it would be deduced that the ratio of the ſides of any two ſquares is given, and the ratio of the diameters of any two circles, &c.   And it is to be obſerved, that the moderns frequently take given ratios, and ratios that are always the ſame, for one and the ſame thing ; and Sir Iſaac Newton has fallen into this miſtake in the 17th Lemma of his Principia, ed. 1713, and in other places ; but this ſhould be carefully avoided, as it may lead into other errors.

P R O P. XIV. XV.

Euclid in this book has ſeveral propoſitions concerning magnitudes, the exceſs of one of which above a given magnitude

nitude has a given ratio to the other; but he has given none
concerning magnitudes whereof one together with a given
magnitude has a given ratio to the other; though these last oc-
cur as frequently in the solution of problems as the first; the
reason of which is, that the last may be all demonstrated by
help of the first; for, if a magnitude, together with a given
magnitude has a given ratio to another magnitude, the excess of
this other above a given magnitude shall have a given ratio to
the first, and on the contrary; as we have demonstrated in prop.
14. And for a like reason prop. 15. has been added to the
data. One example will make the thing clear; suppose it
were to be demonstrated, that if a magnitude A together with a
given magnitude has a given ratio to another magnitude B,
that the two magnitudes A and B, together with a given mag-
nitude, have a given ratio to that other magnitude B; which is
the same proposition with respect to the last kind of magnitudes
above mentioned, that the first part of prop. 16. in this edition
is in respect of the first kind: This is shewn thus; from the hy-
pothesis, and by the first part of prop. 14. the excess of B above
a given magnitude has unto A a given ratio; and, therefore, by
the first part of prop. 17. the excess of B above a given magni-
tude has unto B and A together a given ratio; and by the se-
cond part of prop. 14. A and B together with a given magni-
tude has unto B a given ratio; which is the thing that was to
be demonstrated. In like manner, the other propositions con-
cerning the last kind of magnitudes may be shewn.

PROP. XVI. XVII.

In the third part of prop. 10. in the Greek text, which is the
16th in this edition, after the ratio of EC to CB has been
shown to be given; from this, by inversion and conversion the
ratio of BC to BE is demonstrated to be given; but without
these two steps, the conclusion should have been made only by
citing the 6th proposition. And in like manner, in the first
part of prop. 11. in the Greek, which in this edition is the 17th
from the ratio of DB to BC being given, the ratio of DC to DB
is shewn to be given, by inversion and composition, instead of
citing prop. 7. and the same fault occurs in the second part of
the same prop. 11.

PROP.

P R O P. XXI. XXII.

Thefe now are added, as being wanting to complete the fubject treated of in the four preceding propofitions.

P R O P. XXIII.

This, which is prop. 20. in the Greek text, was feparated from prop. 14. 15. 16. in that text, after which it fhould have been immediately placed, as being of the fame kind ; it is now put into its proper place ; but prop. 21. in the Greek is left out, as being the fame with prop. 14. in that text, which is here prop. 18.

P R O P. XXIV.

This, which is prop. 13. in the Greek, is now put into its proper place, having been disjoined from the three following it in this edition, which are of the fame kind.

P R O P. XXVIII.

This, which in the Greek text is prop. 25. and feveral of the following propofitions are there deduced from def. 4. which is not fufficient, as has been mentioned in the note on that definition : They are therefore now fhewn more explicitly.

P R O P. XXXIV. XXXVI.

Each of thefe has a determination, which is now added, which occafions a change in their demonftrations.

P R O P. XXXVII. XXXIX. XL. XLI.

The 35th and 36th propofitions in the Greek text are joined into one, which makes the 39th in this edition, becaufe the fame enunciation and demonftration ferves both : And for the fame reafon prop. 37. 38. in the Greek are joined into one, which here is the 40.

Prop. 37. is added to the Data, as it frequently occurs in the folution of problems ; and prop. 41. is added to complete the reft.

P R O P. XLII.

This is prop. 39. in the Greek text, where the whole conftruction of prop. 22. of book I. of the elements is put, without need, into the demonftration, but is now only cited.

P R O P. XLV.

This is prop. 42. in the Greek, where the three ftraight lines made ufe of in the conftruction are faid, but not fhewn, to be fuch that any two of them is greater than the third, which is now done.

P R O P.

PROP. XLVII.

This is prop. 44. in the Greek text; but the demonstration of it is changed into another wherein the several cases of it are shewn, which, though necessary, is not done in the Greek.

PROP. XLVIII.

There are two cases in this proposition, arising from the two cases of the third part of prop. 47. on which the 48th depends; and in the composition these two cases are explicitly given.

PROP. LII.

The construction and demonstration of this, which is prop. 48. in the Greek, are made something shorter than in that text.

PROP. LIII.

Prop. 63. in the Greek text is omitted, being only a case of prop. 49. in that text, which is prop. 53. in this edition.

PROP. LVIII.

This is not in the Greek text, but its demonstration is contained in that of the first part of prop. 54. in that text; which proposition is concerning figures that are given in species: This 58th is true of similar figures, though they be not given in species, and, as it frequently occurs, it was necessary to add it.

PROP. LIX. LXI.

This is the 54th in the Greek; and the 77th in the Greek, being the very same with it, is left out, and a shorter demonstration is given of prop. 61.

PROP. LXII.

This, which is most frequently useful, is not in the Greek, and is necessary to prop. 87. 88. in this edition, as also, though not mentioned, to prop. 86. 87. in the former editions. Prop. 66. in the Greek text is made a corollary to it.

PROP. LXIV.

This contains both prop. 74. and 73. in the Greek text; the first case of the 74th is a repetition of prop. 56. from which it is separated in that text by many propositions; and as there is no order in these propositions, as they stand in the Greek, they are now put into the order which seemed most convenient and natural.

The

The demonſtration of the firſt part of prop. 73. in the Greek is groſsly vitiated. Dr Gregory ſays, that the ſentences he has incloſed betwixt two ſtars are ſuperfluous, and ought to be cancelled ; but he has not obſerved, that what follows them is abſurd, being to prove that the ratio [See his figure] of AΓ to ΓK is given, which by the hypotheſis at the beginning of the propoſition is expreſsly given ; ſo that the whole of this part was to be altered, which is done in this prop. 64.

P R O P. LXVII. LXVIII.

Prop. 70. in the Greek text is divided into theſe two, for the ſake of diſtinctneſs ; and the demonſtration of the 67th is rendered ſhorter than that of the firſt part of prop. 70. in the Greek, by means of prop. 23. of book 6. of the elements.

P R O P. LXX.

This is prop. 62. in the Greek text ; prop. 78. in that text is only a particular caſe of it, and is therefore omitted.

Dr Gregory, in the demonſtration of prop. 62. cites the 49th prop. dat. to prove that the ratio of the figure AEB to the parallelogram AH is given ; whereas this was ſhewn a few lines before : And beſides, the 49th prop. is not applicable to theſe two figures ; becauſe AH is not given in ſpecies, but is by the ſtep for which the citation is brought, proved to be given in ſpecies.

P R O P. LXXIII.

Prop. 83. in the Greek text is neither well enunciated nor demonſtrated. The 73d, which in this edition is put in place of it, is really the ſame, as will appear by conſidering [See Dr Gregory's Edition] that A, B, Γ, E in the Greek text are four proportionals ; and that the propoſition is to ſhew that Δ, which has a given ratio to E, is to Γ, as B is to a ſtraight line to which A has a given ratio ; or, by inverſion, that Γ is to Δ, as a ſtraight line to which A has a given ratio is to B ; that is, if the proportionals be placed in this order, viz. Γ, E, A, B, that the firſt Γ is to Δ to which the ſecond E has a given ratio, as a ſtraight line to which the third A has a given ratio is to the fourth B ; which is the enunciation of this 73d, and was thus changed that it might be made like to that of prop. 72. in this edition, which is

the

the 82d in the Greek text : And the demonstration of prop. 73. is the same with that of prop. 72. only making use of prop. 23. instead of prop. 22. of book 5. of the Elements.

PROP. LXXVII.

This is put in place of prop. 79. in the Greek text, which is not a datum, but a theorem premised as a lemma to prop. 80. in that text : And prop. 79. is made cor. 1. to prop. 77. in this edition. Cl. Hardy, in his edition of the data, takes notice, that in prop. 80. of the Greek text, the parallel KL in the figure of prop. 77. in this edition, must meet the circumference, but does not demonstrate it, which is done here at the end of cor. 3. of prop. 77. in the construction for finding a triangle similar to ABC.

PROP. LXXVIII.

The demonstration of this, which is prop. 80. in the Greek, is rendered a good deal shorter by help of prop. 77.

PROP. LXXIX. LXXX. LXXXI.

These are added to Euclid's data, as propositions which are often useful in the solution of problems.

PROP. LXXXII.

This, which is prop. 60. in the Greek text, is placed before the 83d and 84th, which in the Greek are the 58th and 59th, because the demonstration of these two in this edition are deduced from that of prop. 82. from which they naturally follow.

PROP. LXXXVIII. XC.

Dr Gregory, in his preface to Euclid's works, which he published at Oxford in 1703, after having told that he had supplied the defects of the Greek text of the data in innumerable places from several manuscripts, and corrected Cl. Hardy's translation by Mr Bernard's, adds, that the 86th theorem, " or proposition," seemed to be remarkably vitiated, but which could not be restored by help of the manuscripts; then he gives three different translations of it in Latin, according to which he thinks it may be read; the two first have no distinct meaning, and the third, which he says is the best, though it contains a true proposition, which is the 90th in this e-

dition, has no connection in the least with the Greek text. And it is strange that Dr Gregory did not observe, that, if prop. 86, was changed into this, the demonstration of the 86th must be cancelled, and another put in its place: But the truth is, both the enunciation and the demonstration of prop. 86. are quite entire and right, only prop. 87. which is more simple, ought to have been placed before it; and the deficiency which the Doctor justly observes to be in this part of Euclid's data, and which, no doubt, is owing to the carelessness and ignorance of the Greek editors, should have been supplied, not by changing prop. 86. which is both entire and necessary, but by adding the two propositions, which are the 88th and 90th in this edition.

P R O P. XCVIII. C.

These were communicated to me by two excellent geometers, the first of them by the Right Honourable the Earl of Stanhope, and the other by Dr Matthew Stewart; to which I have added the demonstrations.

Though the order of the propositions has been in many places changed from that in former editions, yet this will be of little disadvantage, as the antient geometers never cite the data, and the moderns very rarely.

A S that part of the composition of a problem which is its construction may not be so readily deduced from the analysis by beginners: For their sake the following example is given, in which the derivation of the several parts of the construction from the analysis is particularly shown, that they may be assisted to do the like in other problems.

P R O B L E M.

Having given the magnitude of a parallelogram, the angle of which ABC is given, and also the excess of the square of its side BC above the square of the side AB; to find its sides, and describe it.

The analysis of this is the same with the demonstration of the 87th prop. of the data, and the construction that is given of the problem at the end of that proposition is thus derived from the analysis.

Let

Let EFG be equal to the given angle ABC, and becaufe
in the analyfis it is faid that the ratio of the rectangle AB,
BC to the parallelogram AC is given by the 62d prop. dat.
therefore, from a point in FE, the perpendicular EG is drawn
to FG, as the ratio of FE to EG is the ratio of the rectangle
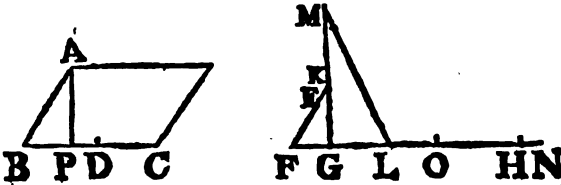
AB, BC to the parallelogram AC by what is fhown at the end
of prop. 62. Next, the magnitude of AC is exhibited by
making the rectangle EG, GH equal to it; and the given
excefs of the fquare of BC above the fquare of BA, to which
excefs the rectangle CB, BD is equal, is exhibited by the
rectangle HG, GL: Then in the analyfis, the rectangle AB
BC is faid to be given, and this is equal to the rectangle FE,
GH, becaufe the rectangle AB, BC is to the parallelogram AC,
as (FE to EG, that is, as the rectangle) FE, GH to EG, GH;
and the parallelogram AC is equal to the rectangle EG, GH,
therefore the rectangle AB, BC, is equal to FE, GH: And
confequently the ratio of the rectangle CB, BD, that is, of the
rectangle HG, GL, to AB, BC, that is, of the ftraight line
DB to BA, is the fame with the ratio (of the rectangle GL,
GH to FE, GH, that is) of the ftraight line GL to FE, which
ratio of DB to BA is the next thing faid to be given in the
analyfis: From this it is plain that the fquare of FE is to the
fquare of GL, as the fquare of BA, which is equal to the
rectangle BC, CD, is to the fquare of BD: The ratio of which
fpaces is the next thing faid to be given: And from this it
follows that four times the fquare of FE is to the fquare of GL,
as four times the rectangle BC, CD is to the fquare of BD;
and, by compofition, four times the fquare of FE together
with the fquare of GL, is to the fquare of GL, as four times
the rectangle BC, CD together with the fquare of BD, is to
the fquare of BD, that is (8. 6.) as the fquare of the
ftraight lines BC, CD taken together is to the fquare of BD,
which ratio is the next thing faid to be given in the analyfis:
And becaufe four times the fquare of FE and the fquare of
GL are to be added together; therefore in the perpendicular
EG

EG there is taken KG equal to FE, and MG equal to the double of it, becaufe thereby the fquares of MG, GL, that is, joining ML, the fquare of ML is equal to four times the fquare of FE and to the fquare of GL: And becaufe the fquare of ML is to the fquare of GL, as the fquare of the ftraight line made up of BC and CD is to the fquare of BD, therefore (22. 6.) ML is to LG, as BC together with CD is to BD ; and, by compofition, ML and LG together, that is, producing GL to N, fo that ML be equal to LN, the ftraight line NG is to GL, as twice BC is to BD ; and by taking GO equal to the half of NG, GO is to GL, as BC to BD, the ratio of which is faid to be given in the analyfis : And from this it follows, that the rectangle HG, GO is to HG, GL, as the fquare of BC is to the rectangle CB, BD which is equal to the rectangle HG, GL; and therefore the fquare of BC is equal to the rectangle HG, GO ; and BC is confequently found by taking a mean proportional betwixt HG and GO, as is faid in the conftruction : And becaufe it was fhown that GO is to GL, as BC to BD, and that now the three firft are found, the fourth BD is found by 12. 6. It was likewife fhown that LG is to FE, or GK, as DB to BA, and the three firft are now found, and thereby the fourth BA. Make the angle ABC equal to EFG, and complete the parallelogram of which the fides are AB, BC, and the conftruction is finifhed ; the reft of the compofition contains the demonftration.

AS the propofitions from the 13th to the 28th may be thought by beginners to be lefs ufeful than the reft, becaufe they cannot fo readily fee how they are to be made ufe of in the folution of problems ; on this account the two following problems are added, to fhow that they are equally ufeful with the other propofitions, and from which it may be eafily judged that many other problems depend upon thefe propofitions.

PROBLEM I.

TO find three ftraight lines fuch, that the ratio of the firft to the fecond is given ; and if a given ftraight line be taken from the fecond, the ratio of the remainder to the third is given ; alfo the rectangle contained by the firft and third is given,

Let AB be the firſt ſtraight line, CD the ſecond, and EF the
third : And becauſe the ratio of AB to CD is given, and that
if a given ſtraight line be taken from CD, the ratio of the re-
mainder to EF is given ; therefore[a] the exceſs of the firſt AB   [a] 44. dat.
above a given ſtraight line has a given ratio to the third EF :
Let BH be that given ſtraight line ; therefore AH, the exceſs
of AB above it, has a given ratio to EF ; and

A   H   B

conſequently[b] the rectangle BA, AH, has a   [b] 1. 9.
given ratio to the rectangle AB, EF, which

C   G   D

laſt rectangle is given by the hypotheſis ;
therefore[c] the rectangle BA, AH is given,   E   F   [c] 2. dat.
and BH the exceſs of its ſides is given ; where-
fore the ſides AB, AH are given[d] : And be-

K   NML O   [d] 85. dat.

cauſe the ratios of AB to CD, and of AH to
EF are given, CD and EF are[e] given.

The Compoſition.

Let the given ratio of KL to KM be that which AB is requi-
red to have to CD ; and let DG be the given ſtraight line which
is to be taken from CD, and let the given ratio of KM to KN
be that which the remainder muſt have to EF ; alſo let the gi-
ven rectangle NK, KO be that to which the rectangle AB, EF
is required to be equal : Find the given ſtraight line BH which
is to be taken from AB, which is done, as plainly appears from
prop. 24. dat. by making as KM to KL, ſo GD to HB.   To
the given ſtraight line BH apply[e] a rectangle equal to LK, KO e 29. 6.
exceeding by a ſquare, and let BA, AH be its ſides : Then is
AB the firſt of the ſtraight lines required to be found, and by
making as LK to KM, ſo AB to DC, DC will be the ſecond :
And laſtly, make as KM to KN, ſo CG to EF, and EF is the
third.
For as AB to CD, ſo is HB to GD, each of theſe ratios be-
ing the ſame with the ratio of LK to KM ; therefore[f] AH is f 19. 5.
to CG, as (AB to CD, that is, as) LK to KM ; and as CG
to EF, ſo is KM to KN ; wherefore, ex aequali, as AH to EF,
ſo is LK to KN : And as the rectangle BA, AH to the rectan-
gle BA, EF, ſo is[g] the rectangle LK, KO to the rectangle KN, g 1. 6.
KO : And, by the conſtruction, the rectangle BA, AH is equal
to LK, KO : Therefore[h] the rectangle AB, EF is equal to the h 14. 5.
given rectangle NK, KO : And AB has to CD the given ratio of
KL to KM ; and from CD the given ſtraight line GD being taken,
the remainder CG has to EF the given ratio of KM to KN.
Q. E. D.

                                 P R O B.

PROB. II.

TO find three ſtraight lines ſuch, that the ratio of the firſt to the ſecond is given ; and if a given ſtraight line be taken from the ſecond, the ratio of the remainder to the third is given ; alſo the ſum of the ſquares of the firſt and third is given.

Let AB be the firſt ſtraight line, BC the ſecond, and BD the third : And becauſe the ratio of AB to BC is given, and that if a given ſtraight line be taken from BC, the ratio of the remainder to BD is given ; therefore ᵃ the exceſs of the firſt AB above a given ſtraight line, has a given ratio to the third BD : Let AE be that given ſtraight line, therefore the remainder EB has a given ratio to BD : Let BD be placed at right angles to EB, and join DE ; then the triangle EBD is ᵇ given in ſpecies ; wherefore the angle BED is given : Let AE, which is given in magnitude, be given alſo in poſition, as alſo the point E, and the ſtraight line ED will be given ᶜ in poſition : Join AD, and becauſe the ſum of the ſquares of AB, BD, that is ᵈ, the ſquare of AD is given, therefore the ſtraight line AD is given in magnitude ; and it is alſo given ᵉ in poſition, becauſe from the given point A it is drawn to the ſtraight line ED given in poſition : Therefore the point D, in which the two ſtraight lines AD, ED given in poſition cut one another, is given ᶠ : And the ſtraight line DB which is at right angles to AB is given ᵍ in poſition, and AB is given in poſition, therefore ᶠ the point B is given : And the points A, D are given, wherefore ʰ the ſtraight lines AB, BD are given : And the ratio of AB to BC is given, and therefore ⁱ BC is given.

a 24. dat.
b 44. dat.
c 32. dat.
d 47. 1.
e 34. dat.
f 28. dat.
g 33. dat.
h 29. dat.
i 2. dat.

The Compoſition.

Let the given ratio of FG to GH be that which AB is required to have to BC, and let HK be the given ſtraight line which is to be taken from BC, and let the ratio which the re-

A   E   B N M   C   F   G   H   K

mainder is required to have to BD be the given ratio of HG to GL, and place GL at right angles to FH, and join LF, LH:
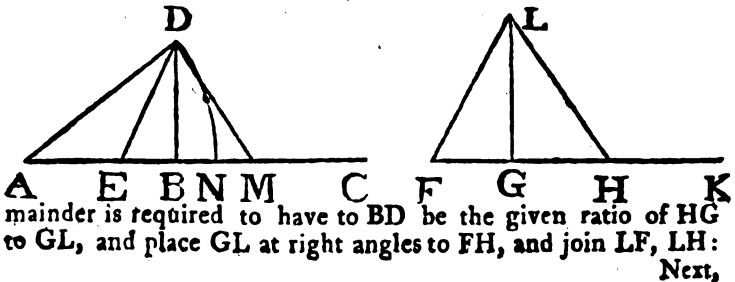
Next,

Next, as HG is to GF, so make HK to AE; produce AE to
N, so that AN be the straight line to the square of which the
sum of the squares of AB, BD is required to be equal; and
make the angle NED equal to the angle GFL; and from the
centre A at the distance AN describe a circle, and let its cir-
cumference meet ED in D, and draw DB perpendicular to AN,
and DM making the angle BDM equal to the angle GLH.
Lastly, produce BM to C, so that MC be equal to HK; then is
AB the first, BC the second, and BD the third of the straight
lines that were to be found.

For the triangles EBD, FGL, as also DBM, LGH being
equiangular, as EB to BD, so is FG to GL; and as DB to
BM, so is LG to GH; therefore, ex aequali, as EB to BM,
so is (FG to GH, and so is) AE to HK or MC; wherefore [k], k 12. 5.
AB is to BC, as AE to HK, that is, as FG to GH, that is,
in the given ratio; and from the straight line BC taking MC,
which is equal to the given straight line HK, the remainder
BM has to BD the given ratio of HG to GL; and the sum of
the squares of AB, BD is equal [d] to the square of AD or AN, d 47. 1.
which is the given space. Q. E. D.

I believe it would be in vain to try to deduce the preceding
construction from an algebraical solution of the problem.

F I N I S.

THE

ELEMENTS

OF

PLANE AND SPHERICAL

TRIGONOMETRY.

PLANE TRIGONOMETRY.

LEMMA I. Fig. 1.

LET ABC be a rectilineal angle, if about the point B as a centre, and with any distance BA, a circle be described, meeting BA, BC, the straight lines including the angle ABC in A, C; the angle ABC will be to four right angles, as the arch AC to the whole circumference.

Produce AB till it meet the circle again in F, and through B draw DE perpendicular to AB, meeting the circle in D, E.

By 33. 6. Elem. the angle ABC is to a right angle ABD, as the arch AC to the arch AD; and quadrupling the consequents, the angle ABC will be to four right angles, as the arch AC to four times the arch AD, or to the whole circumference.

LEMMA II. Fig. 2.

LET ABC be a plane rectilineal angle as before: About B as a centre with any two distances BD, BA, let two circles be described meeting BA, BC in D, E, A, C; the arch AC will be to the whole circumference of which it is an arch, as the arch DE is to the whole circumference of which it is an arch.

By Lemma 1. the arch AC is to the whole circumference of which it is an arch, as the angle ABC is to four right angles; and by the same Lemma 1. the arch DE is to the whole circumference of which it is an arch, as the angle ABC is to four right angles; therefore the arch AC is to the whole circumference of which it is an arch, as the arch DE to the whole circumference of which it is an arch.

DEFINITIONS. Fig. 3.

I.

LET ABC be a plane rectilineal angle; if about B as a centre, with BA any distance, a circle ACF be described meeting BA, BC, in A, C; the arch AC is called the measure of the angle ABC.

II.

The circumference of a circle is supposed to be divided into

360 equal parts called degrees, and each degree into 60 equal
parts called minutes, and each minute into 60 equal parts
called feconds, &c. And as many degrees, minutes, fe-
conds, &c. as are contained in any arch, of fo many degrees,
minutes, feconds, &c. is the angle, of which that arch is the
meafure, faid to be.

COR. Whatever be the radius of the circle of which the mea-
fure of a given angle is an arch, that arch will contain the
fame number of degrees, minutes, feconds, &c. as is mani-
feft from Lemma 2.

III.

Let AB be produced till it meet the circle again in F, the angle
CBF, which, together with ABC, is equal to two right angles,
is called the *Supplement* of the angle ABC.

IV.

A ftraight line CD drawn through C, one of the extremities of
the arch AC, perpendicular upon the diameter paffing through
the other extremity A, is called the *Sine* of the arch AC, or
of the angle ABC, of which it is the meafure.

COR. The *Sine* of a quadrant, or of a right angle, is equal to
the radius.

V.

The fegment DA of the diameter paffing through A, one ex-
tremity of the arch AC between the line CD, and that ex-
tremity, is called the *Verfed Sine* of the arch AC, or angle
ABC.

VI.

A ftraight line AE touching the circle at A, one extremity of
the arch AC, and meeting the diameter BC paffing through
the other extremity C in E, is called the *Tangent* of the arch
AC, or of the angle ABC.

VII.

The ftraight line BE between the centre and the extremity of
the tangent AE, is called the *Secant* of the arch AC, or
angle ABC.

COR. to def. 4. 6. 7. The fine, tangent, and fecant of any angle
ABC, are likewife the fine, tangent, and fecant of its fupple-
ment CBF.

It is manifeft from Def. 4. that CD is the fine of the angle
CBF. Let CB be produced till it meet the circle again in
G.; and it is manifeft that AE is the tangent, and BE the
fecant, of the angle ABG or EBF, from def. 6. 7.

COR.

Fig. 4.

Cor. to Def. 4. 5. 6. 7. The fine, verfed fine, tangent, and fe-
cant, of any arch which is the meafure of any given angle
ABC, is to the fine, verfed fine, tangent, and fecant, of any
other arch which is the meafure of the fame angle, as the
radius of the firft is to the radius of the fecond.

Let AC, MN be meafures of the angles ABC, according to
def. 1. CD the fine, DA the verfed fine, AE the tangent,
and BE the fecant of the arch AC, according to def. 4. 5.
6. 7. and NO the fine, OM the verfed fine, MP the tan-
gent, and BP the fecant of the arch MN, according to the
fame definitions. Since CD, NO, AE, MP are parallel,
CD is to NO as the radius CB to the radius NB, and AE
to MP as AB to BM, and BC or BA to BD as BN or BM
to BO; and, by converfion, DA to MO as AB to MB.
Hence the corollary is manifeft; therefore, if the radius
be fuppofed to be divided into any given number of equal
parts, the fine, verfed fine, tangent, and fecant of any given
angle, will each contain a given number of thefe parts; and,
by trigonometrical tables, the length of the fine, verfed fine,
tangent, and fecant of any angle may be found in parts of
which the radius contains a given number; and, vice verfa,
a number expreffing the length of the fine, verfed fine, tan-
gent, and fecant being given, the angle of which it is the
fine, verfed fine, tangent, and fecant may be found.

VIII.

Fig. 3.

The difference of an angle from a right angle is called the
complement of that angle. Thus, it BH be drawn perpen-
dicular to AB, the angle CBH will be the complement of the
angle ABC, or of CBF.

IX.

Let HK be the tangent, CL or DB, which is equal to it,
the fine, and BK the fecant of CBH, the complement of
ABC, according to def. 4. 6. 7. HK is called the *co-tan-
gent*, BD the *co-fine*, and BK the *co-fecant* of the angle
ABC.

Cor. 1. The radius is a mean proportional between the tan-
gent and co-tangent.

For, fince HK, BA are parallel, the angles HKB, ABC will
be equal, and the angles KHB, BAE are right; therefore

the triangles BAE, KHB are fimilar, and therefore AE is to AB, as BH or BA to HK.

COR. 2. The radius is a mean proportional between the co-fine and fecant of any angle ABC.

Since CD, AE are parallel, BD is to BC or BA, as BA to BE.

PROP. I. FIG. 5.

IN a right angled plain triangle, if the hypothenufe be made radius, the fides become the fines of the angles oppofite to them; and if either fide be made radius, the remaining fide is the tangent of the angle oppofite to it, and the hypothenufe the fecant of the fame angle.

Let ABC be a right angled triangle; if the hypothenufe BC be made radius, either of the fides AC will be the fine of the angle ABC oppofite to it; and if either fide BA be made radius, the other fide AC will be the tangent of the angle ABC oppofite to it, and the hypothenufe BC the fecant of the fame angle.

About B as a centre, with BC, BA for diftances, let two circles CD, EA be defcribed, meeting BA, BC in D, E: Since CAB is a right angle, BC being radius, AC is the fine of the angle ABC by def. 4. and BA being radius, AC is the tangent, and BC the fecant of the angle ABC, by def. 6. 7.

COR. 1. Of the hypothenufe a fide and an angle of a right angled triangle, any two being given, the third is alfo given.

COR. 2. Of the two fides and an angle of a right angled triangle, any two being given, the third is alfo given.

PROP. II. FIG. 6. 7.

THE fides of a plain triangle are to one another, as the fines of the angles oppofite to them.

In right angled triangles, this prop. is manifeft from prop. 1. for if the hypothenufe be made radius, the fides are the fines of the angles oppofite to them, and the radius is the fine of a right angle (cor. to def. 4) which is oppofite to the hypothenufe.

In

In any oblique angled triangle ABC, any two fides AB, AC will be to one another as the fines of the angles ACB, ABC which are oppofite to them.

From C, B draw CE, BD perpendicular upon the oppofite fides AB, AC produced, if need be. Since CEB, CDB are right angles, BC being radius, CE is the fine of the angle CBA, and BD the fine of the angle ACB; but the two triangles CAE, DAB have each a right angle at D and E; and likewife the common angle CAB; therefore they are fimilar, and confe-quently, CA is to AB, as CE to DB; that is, the fides are as the fines of the angles oppofite to them.

Cor. Hence of two fides, and two angles oppofite to them, in a plain triangle, any three being given, the fourth is alfo given.

P R O P. III. Fig. 8.

IN a plain triangle, the fum of any two fides is to their difference, as the tangent of half the fum of the angles at the bafe, to the tangent of half their difference.

Let ABC be a plain triangle, the fum of any two fides AB, AC will be to their difference as the tangent of half the fum of the angles at the bafe ABC, ACB to the tangent of half their difference.

About A as a centre, with AB the greater fide for a diftance, let a circle be defcribed, meeting AC produced in E, F, and BC in D; join DA, EB, FB; and draw FG parallel to BC, meeting EB in G.

The angle EAB (32. 1.) is equal to the fum of the angles at the bafe, and the angle EFB at the circumference is equal to the half of EAB at the centre (20. 3.); therefore EFB is half the fum of the angles at the bafe; but the angle ACB (32. 1.) is equal to the angles CAD and ADC, or ABC together; therefore FAD is the difference of the angles at the bafe, and FBD at the circumference, or BFG, on account of the parallels FG, BD, is the half of that difference; but fince the angle EBF in a femicircle is a right angle (1. of this) FB being radius, BE, BG, are the tangents of the angles EFB, BFG; but it is manifeft that EC is the fum of the fides BA, AC, and CF their difference; and fince BC, FG are parallel (2. 6.) EC is to CF, as EB to BG; that is, the fum of the

H h 3

fides

fides is to their difference, as the tangent of half the fum of the angles at the bafe to the tangent of half their difference.

PROP. IV. Fig. 18.

IN any plain triangle BAC, whofe two fides are BA, AC and bafe BC, the lefs of the two fides, which let be BA, is to the greater AC as the radius is to the tangent of an angle, and the radius is to the tangent of the excefs of this angle above half a right angle as the tangent of half the fum of the angles B and C at the bafe, is to the tangent of half their difference.

At the point A, draw the ftraight line EAD perpendicular to BA; make AE, AF, each equal to AB, and AD to AC; join BE, BF, BD, and from D, draw DG perpendicular upon BF. And becaufe BA is at right angles to EF, and EA, AB, AF are equal, each of the angles EBA, ABF is half a right angle, and the whole EBF is a right angle; alfo (4. 1. El.) EB is equal to BF. And fince EBF, FGD are right angles, EB is parallel to GD, and the triangles EBF, FGD are fimilar; therefore EB is to BF as DG to GF, and EB being equal to BF, FG muft be equal to GD. And becaufe BAD is a right angle, BA the lefs fide is to AD or AC the greater, as the radius is to the tangent of the angle ABD; and becaufe BGD is a right angle, BG is to GD or GF as the radius is to the tangent of GBD, which is the excefs of the angle ABD above ABF half a right angle. But becaufe EB is parallel to GD, BG is to GF as ED is to DF, that is, fince ED is the fum of the fides BA, AC and FD their difference, (3. of this), as the tangent of half the fum of the angles B, C, at the bafe to the tangent of half their difference. Therefore, in any plain triangle, &c. Q. E. D.

PROP. V. Fig. 9. and 10.

IN any triangle, twice the rectangle contained by any two fides is to the difference of the fum of the fquares of thefe two fides, and the fquare of the bafe, as the radius is to the co-fine of the angle included by the two fides.

Let ABC be a plain triangle, twice the rectangle ABC contained by any two fides BA, BC is to the difference of the fum
of

of the fquares of BA, BC, and the fquare of the bafe AC, as the radius to the co-fine of the angle ABC.

From A, draw AD perpendicular upon the oppofite fide BC, then (by 12. and 13. 2. El.) the difference of the fum of the fquares of AB, BC, and the fquare of the bafe AC, is equal to twice the rectangle CBD ; but twice the rectangle CBA is to twice the rectangle CBD, that is, to the difference of the fum of the fquares of AB, BC, and the fquare of AC, (1. 6.) as AB to BD ; that is, by prop. 1. as radius to the fine of BAD, which is the complement of the angle ABC, that is, as radius to the co-fine of ABC.

PROP. VI. FIG. 11.

IN any triangle ABC, whofe two fides are AB, AC, and bafe BC, the rectangle contained by half the peri-meter, and the excefs of it above the bafe BC, is to the rectangle contained by the ftraight lines, by which the half of the perimeter exceeds the other two fides AB, AC, as the fquare of the radius is to the fquare of the tangent of half the angle BAC oppofite to the bafe.

Let the angles BAC, ABC be bifected by the ftraight lines AG, BG; and, producing the fide AB, let the exterior angle CBH be bifected by the ftraight line BK, meeting AG in K; and from the points G, K, let there be drawn perpendicular upon the fides the ftraight lines GD, GE, GF, KH, KL, KM. Since therefore (4. 4.) G is the centre of the circle in-fcribed in the triangle ABC, GD, GF, GE will be equal, and AD will be equal to AE, BD to BF, and CE to CF. In like manner KH, KL, KM will be equal, and BH will be equal to BM, and AH to AL, becaufe the angles HBM, HAL are bi-fected by the ftraight lines BK, KA : And becaufe in the tri-angles KCL, KCM, the fides LK, KM are equal, KC is com-mon and KLC, KMC are right angles, CL will be equal to CM : Since therefore BM is equal to BH, and CM to CL ; BC will be equal to BH and CL together ; and, adding AB and AC together, AB, AC, and BC will together be equal to AH and AL together : But AH, AL are equal : Wherefore each of them is equal to half the perimeter of the triangle ABC : But fince AD, AE are equal, and BD, BF, and alfo CE, CF, AB together with FC, will be equal to half the pe-rimeter of the triangle to which AH or AL was fhewn to be equal ; taking away therefore the common AB, the remain-

der FC will be equal to the remainder BH : In the same man-
ner is it demonstrated, that BF is equal to CL : And since the
point B, D, G, F, are in a circle, the angle DGF will be e-
qual to the exterior and opposite angle FBH, (22. 3.) ; where-
fore their halves BGD, HBK will be equal to one another :
The right angled triangles BGD, HBK will therefore be e-
quiangular, and GD will be to BD, as BH to HK, and the
rectangle contained by GD, HK will be equal to the rectangle
DBH or BFC : But since AH is to HK, as AD to DG, the
rectangle HAD (22. 6.) will be to the rectangle contained by
HK, DG, or the rectangle BFC, (as the square of AD is to
the square of DG, that is) as the square of the radius to the
square of the tangent of the angle DAG, that is, the half of
BAC : But HA is half the perimeter of the triangle ABC, and
AD is the excess of the same above HD, that is, above the
base BC ; but BF or CL is the excess of HA or AL above the
side AC, and FC, or HB is the excess of the same HA above
the side AB ; therefore the rectangle contained by half the pe-
rimeter, and the excess of the same above the base, viz. the
rectangle HAD, is to the rectangle contained by the straight
lines by which the half of the perimeter exceeds the other two
sides, that is, the rectangle BFC, as the square of the radius
is to the square of the tangent of half the angle BAC opposite
to the base.   Q. E. D.

PROP. VII. FIG. 12. 13.

IN a plain triangle, the base is to the sum of the sides,
as the difference of the sides is to the sum or differ-
ence of the segments of the base made by the perpendi-
cular upon it from the vertex, according as the square of
the greater side is greater or less than the sum of the
squares of the lesser side and the base.

Let ABC be a plane triangle ; if from A the vertex be drawn
a straight line AD perpendicular upon the base BC, the base
BC will be to the sum of the sides BA, AC, as the difference
of the same sides is to the sum or difference of the segments CD,
BD, according as the square of AC the greater side is greater
or less than the sum of the squares of the lesser side AB, and the
base BC.

About A as a centre, with AC the greater side for a di-
stance, let a circle be described meeting AB produced in E,
F, and CB in G : It is manifest that FB is the sum, and BE

the

the difference of the fides; and fince AD is perpendicular to GC, GD, CD will be equal; confequently GB will be equal to the fum or difference of the fegments CD, BD, according as the perpendicular AD meets the bafe, or the bafe produced; that is, (by Conv. 12. and 13. 2.) according as the fquare of AC is greater or lefs than the fum of the fquares of AB, BC: But (by 35. 3.) the rectangle CBG is equal to the rectangle EBF; that is, (16. 6.) BC is to BF, as BE is to BG: that is, the bafe is to the fum of the fides, as the difference of the fides is to the fum or difference of the fegments of the bafe made by the perpendicular from the vertex, according as the fquare of the greater fide is greater or lefs than the fum of the fquares of the leffer fide and the bafe. Q. E. D.

PROP. VIII. PROB. Fig. 14.

THE fum and difference of two magnitudes being given, to find them.

Half the given fum added to half the given difference, will be the greater, and half the difference fubtracted from half the fum, will be the lefs.

For, let AB be the given fum, AC the greater, and BC the lefs. Let AD be half the given fum; and to AD, DB, which are equal, let DC be added, then AC will be equal to BD, and DC together; that is, to BC, and twice DC; confequently twice DC is the difference, and DC half that difference; but AC the greater is equal to AD, DC; that is, to half the fum added to half the difference, and BC the lefs is equal to the excefs of BD, half the fum above DC half the difference. Q. E. F.

SCHOLIUM.

Of the fix parts of a plain triangle (the three fides and three angles) any three being given, to find the other three is the bufinefs of plane trigonometry; and the feveral cafes of that problem may be refolved by means of the preceding propofitions, as in the two following, with the tables annexed. In thefe, the folution is expreffed by a fourth proportional to three given lines; but if the given parts be expreffed by numbers from trigonometrical tables, it may be obtained arithmetically by the common Rule of Three.

SOLUTION

Note. In the tables the following abbreviations are ufed. R, is put for the Radius; T, for Tangent; and S, for Sine. Degrees, minutes, feconds, &c. are written in this manner; 30° 25' 13", &c. which fignifies 30 degrees, 25 minutes, 13 feconds, &c.

SOLUTION of the Cases of right angled Triangles.

GENERAL PROPOSITION.

IN a right angled triangle, of the three fides and three angles, any two being given befides the right angle, the other three may be found, except when the two a-cute angles are given, in which cafe the ratios of the fides are only given, being the fame with the ratios of the fines of the angles oppofite to them.

It is manifeft from 47. 1. that of the two fides and hypothe-nufe any two be given, the third may alfo be found. It is alfo manifeft from 32. 1. that if one of the acute angles of a right-angled triangle be given, the other is alfo given, for it is the complement of the former to a right angle.

If two angles of any triangle be given, the third is alfo given, being the fupplement of the two given angles to two right an-gles.

Fig. 15.   The other cafes may be refolved by help of the preceding pro-pofitions, as in the following table.

	GIVEN.	SOUGHT.	
1	Two fides, AB AC.	The angles B, C.	AB : AC :: R : T, B. of which C is the complement.
2	AB, BC, a fide and the hypothenufe.	The angles B, C.	BC : BA :: R : S, C, of which B is the complement.
3	AB, B, a fide and an angle.	The other fide AC.	R : T, B :: BA : AC.
4	AB and B, a fide and an angle.	The hypo-thenufe BC.	S, C : R :: BA : BC.
5	BC and B, the hypothenufe and an angle.	The fide AC.	R : S, B :: BC : CA.

Thefe five cafes are refolved by prop. 1.

SOLUTION of the Cases of oblique-angled Triangles.

GENERAL PROPOSITION.

IN an oblique-angled triangle, of the three sides and three angles, any three being given, the other three may be found, except when the three angles are given; in which case the ratios of the sides are only given, being the same with the ratios of the sines of the angles opposite to them.

	GIVEN.	SOUGHT.	
1	A, B, and therefore C, and the side AB.	BC, AC.	S, C : S, A :: AB : BC, and also, S, C : S, B :: AB : AC. (2.) Fig. 16. 17.
2	AB, AC, and B, two sides and an angle opposite to one of them.	The angles A and C.	AC : AB :: S, B : S, C. (2.) This case admits of two solutions ; for C may be greater or less than a quadrant. (Cor. to def. 4.)
3	AB, AC, and A, two sides, and the included angle.	The angles B and C.	AB+AC : AB—AC :: T, $\frac{C+B}{2}$: T, $\frac{C-B}{2}$: (3.) and the sum and difference of the angles C, B, being given, each of them is given. (7.) *Otherwise.* FIG. 18. BA:AC:: R : T, ABC, and also R : T, ABC—45° : T, $\frac{B+C}{2}$: T, $\frac{B-C}{2}$: (4.) therefore B and C are given as before. (7.)

GIVEN

	Given.	Sought.	
4	AB, BC, CA, the three sides.	A, B, C, the three angles.	$2 AC \times CB : ACq + CBq - ABq :: R : CoS, C.$ If $ACq + CBq$ be greater than ABq. Fig. 16. $2 AC \times CB : ABq - ACq - CBq :: R : CoS, C.$ If ABq be greater than $ACq + CBq$ Fig. 17. (4.) *Otherwise.* Let $AB + BC + AC = 2P.$ $P \times \overline{P - AB} : \overline{P - AC} \times \overline{P - BC} :: Rq : Tq, \frac{1}{2}C,$ and hence C is known. (5.) *Otherwise.* Let AD be perpendicular to BC. 1. If ABq be less than $ACq + CBq$. FIG. 16. $BC : BA + AC :: BA - AC : BD - DC,$ and BC the sum of BD, DC is given; therefore each of them is given. (7.) 2 If ABq be greater than $ACq + CBq$. Fig. 17. $BC : BA + AC :: BA - AC : BD + DC;$ and BC the difference of BD, DC is given, therefore each of them is given (7.) And $CA : CD :: R : CoS, C.$ (1.) and C being found, A and B are found by case 2. or 3.

SPHERICAL TRIGONOMETRY.

DEFINITIONS.

I.

THE pole of a circle of the fphere is a point in the fuper-
ficies of the fphere, from which all ftraight lines drawn
to the circumference of the circle are equal.

II.

A great circle of the fphere is any whofe plane paffes through
the centre of the fphere, and whofe centre therefore is the
fame with that of the fphere.

III.

A fpherical triangle is a figure upon the fuperficies of a fphere
comprehended by three arches of three great circles, each of
which is lefs than a femicircle.

IV.

A fpherical angle is that which on the fuperficies of a fphere is
contained by two arches of great circles, and is the fame with
the inclination of the planes of thefe great circles.

PROP. I.

GREAT circles bifect one another.

As they have a common centre their common fection will be
a diameter of each which will bifect them.

PROP. II. Fig. 1.

THE arch of a great circle betwixt the pole and the
circumference of another is a quadrant.

Let ABC be a great circle, and D its pole; if a great circle
DC pafs through D, and meet ABC in C, the arch DC will be
a quadrant.
Let the great circle CD meet ABC again in A, and let
AC be the common fection of the great circles, which will
pafs

pafs through E the centre of the fphere : Join DE, DA, DC:
By def. 1. DA, DC are equal, and AE, EC are alfo equal, and
DE is common ; therefore (8. 1.) the angles DEA, DEC are
equal ; wherefore the arches DA, DC are equal, and confe-
quently each of them is a quadrant. Q. E. D.

PROP. III. FIG. 2.

IF a great circle be defcribed meeting two great cir-
cles AB, AC pafling through its pole A in B, C, the
angle at the centre of the fphere upon the circumfe-
rence BC, is the fame with the fpherical angle BAC,
and the arch BC is called the meafure of the fpherical
angle BAC.

Let the planes of the great circles AB, AC interfect one an-
other in the ftraight line AD pafling through D their common
centre ; join DB, DC.
Since A is the pole of BC, AB, AC will be quadrants, and
the angles ADB, ADC right angles ; therefore (6. def. 11.) the
angle CDB is the inclination of the planes of the circles AB,
AC ; that is, (def. 4.) the fpherical angle BAC. Q. E. D.
Cor. If through the point A, two quadrants AB, AC, be
drawn, the point A will be the pole of the great circle BC,
pafling through their extremities B, C.
Join AC, and draw AE a ftraight line to any other point E
in BC ; join DE : Since AC, AB are quadrants, the angles
ADB, ADC are right angles, and AD will be perpendicular to
the plane of BC : Therefore the angle ADE is a right angle,
and AD, DC are equal to AD, DE, each to each ; therefore
AE, AC are equal, and A is the pole of BC, by def. 1.
Q. E. D.

PROP. IV. FIG. 3.

IN ifofceles fpherical triangles, the angles at the bafe are
equal.

Let ABC be an ifofceles triangle, and AC, CB the equal
fides ; the angles BAC, ABC, at the bafe AB, are equal.

Let

Let D be the centre of the fphere, and join DA, DB, DC ; in DA take any point E, from which draw, in the plane ADC, the ftraight line EF at right angles to ED meeting CD in F, and draw, in the plane ADB, EG at right angles to the fame ED ; therefore the rectilineal angle FEG is (6. def. 11.) the inclination of the planes ADC, ADB, and therefore is the fame with the fpherical angle BAC : From F draw FH perpendicular to DB, and from H draw, in the plane ADB, the ftraight line HG at right angles to HD meeting EG in G, and join GF. Becaufe DE is at right angles to EF and EG, it is perpendicular to the plane FEG, (4. 11.) and therefore the plane FEG is perpendicular to the plane ADB, in which DE is : (18. 11.) In the fame manner the plane FHG is perpendicular to the plane ADB ; and therefore GF the common fection of the planes FEG, FHG is perpendicular to the plane ADB ; (19. 11.) and becaufe the angle FHG is the inclination of the planes BDC, BDA, it is the fame with the fpherical angle ABC ; and the fides AC, CB of the fpherical triangle being equal, the angles EDF, HDF, which ftand upon them at the centre of the fphere, are equal ; and in the triangles EDF, HDF the fide DF is common, and the angles DEF, DHF are right angles ; therefore EF, FH are equal ; and in the triangles FEG, FHG the fide GF is common, and the fides EG, GH will be equal by the 47. 1. and therefore the angle FEG is equal to FHG ; (8. 1.) that is, the fpherical angle BAC is equal to the fpherical angle ABC.

<center>P R O P. V. Fig. 3.</center>

IF, in a fpherical triangle ABC, two of the angles BAC, ABC be equal, the fides BC, AC oppofite to them, are equal.

Read the conftruction and demonftration of the preceding propofition, unto the words, " and the fides AC, CB," &c. and the reft of the demonftration will be as follows, viz.

And the fpherical angles BAC, ABC being equal, the rectilineal angles FEG, FHG, which are the fame with them, are equal ; and in the triangles FGE, FGH the angles at G are right angles, and the fide FG oppofite to two of the equal
<center>angles</center>

angles is common; therefore (26. 1.) EF is equal to FH; and in the right angled triangles DEF, DHF the side DF is common; wherefore (47. 1.) ED is equal to DH, and the angles EDF, HDF are therefore equal, (4. 1.) and consequently the sides AC, BC of the spherical triangle are equal.

PROP. VI. Fig. 4.

ANY two sides of a spherical triangle are greater than the third.

Let ABC be a spherical triangle, any two sides AB, BC will be greater than the other side AC.
Let D be the centre of the sphere; join DA, DB, DC.
The solid angle at D is contained by three plane angles ADB, ADC, BDC; and by 20. 11. any two of them ADB, BDC are greater than the third ADC; that is, any two sides AB, BC of the spherical triangle ABC, are greater than the third AC.

PROP. VII. Fig. 4.

THE three sides of a spherical triangle are less than a circle.

Let ABC be a spherical triangle as before, the three sides AB, BC, AC are less than a circle.
Let D be the centre of the sphere: The solid angle at D is contained by three plane angles BDA, BDC, ADC, which together are less than four right angles, (21. 11.) therefore the sides AB, BC, AC together, will be less than four quadrants; that is, less than a circle.

PROP. VIII. Fig. 5.

IN a spherical triangle the greater angle is opposite to the greater side; and conversely.

Let ABC be a spherical triangle, the greater angle A is opposed to the greater side BC.
Let the angle BAD be made equal to the angle B, and then BD, DA will be equal, (5. of this) and therefore AD, DC

DC are equal to BC; but AD, DC are greater than AC, (6. of this), therefore BC is greater than AC, that is, the greater angle A is oppofite to the greater fide BC. The converfe is de-monftrated as prop. 19. 1. El. Q. E. D.

PROP. IX. FIG. 6.

IN any fpherical triangle ABC, if the fum of the fides AB, BC be greater, equal, or lefs than a femicircle, the internal angle at the bafe AC will be greater, equal, or lefs than the external and oppofite BCD; and there-fore the fum of the angles A and ACB will be greater, equal, or lefs than two right angles.

Let AC, AB produced meet in D.
1. If AB, BC be equal to a femicircle, that is, to AD, BC, BD will be equal, that is, (4. of this) the angle D, or the angle A will be equal to the angle BCD.
2. If AB, BC together be greater than a femicircle, that is, greater than ABD, BC will be greater than BD; and therefore (8. of this) the angle D, that is, the angle A, is greater than the angle BCD.
3. In the fame manner is it fhown, that if AB, BC together be lefs than a femicircle, the angle A is lefs than the angle BCD. And fince the angles BCD, BCA are equal to two right angles, if the angle A be greater than BCD, A and ACB together will be greater than two right angles. If A be equal to BCD, A and ACB together will be equal to two right angles; and if A be lefs than BCD, A and ACB will be lefs than two right angles. Q. E. D.

PROP. X. FIG. 7.

IF the angular points A, B, C of the fpherical triangle ABC be the poles of three great circles, thefe great circles by their interfections will form another triangle FDE, which is called fupplemental to the former; that is, the fides FD, DE, EF are the fupplements of the

I i   meafures

measures of the oppofite angles C, B, A, of the triangle ABC, and the meafures of the angles F, D, E of the triangle FDE, will be the fupplements of the fides AC, BC, BA, in the triangle ABC.

Let AB produced meet DE, EF in G, M, and AC meet FD, FE in K, L, and BC meet FD, DE in N, H.

Since A is the pole of FE, and the circle AC paffes through A, EF will pafs through the pole of AC, (13. 15. 1. Th.) and fince AC paffes through C, the pole of FD, FD will pafs through the pole of AC; therefore the pole of AC is in the point F, in which the arches DF, EF interfect each other. In the fame manner, D is the pole of BC, and E the pole of AB.

And fince F, E are the poles of AL, AM, FL and EM are quadrants, and FL, EM together, that is, FE and ML together, are equal to a femicircle. But fince A is the pole of ML, ML is the meafure of the angle BAC, confequently FE is the fupplement of the meafure of the angle BAC. In the fame manner, ED, DF are the fupplements of the meafures of the angles ABC, BCA.

Since likewife CN, BH are quadrants, CN, BH together, that is, NH, BC together are equal to a femicircle; and fince D is the pole of NH, NH is the meafure of the angle FDE, therefore the meafure of the angle FDE is the fupplement of the fide BC. In the fame manner, it is fhown that the meafures of the angles DEF, EFD are the fupplements of the fides AB, AC, in the triangle ABC. Q. E. D.

PROP. XI. Fig. 7.

THE three angles of a fpherical triangle are greater than two right angles, and lefs than fix right angles.

The meafures of the angles A, B, C, in the triangle ABC, together with the three fides of the fupplemental triangle DEF, are 10. of this) equal to three femicircles; but the three fides of the triangle FDE, are (7. of this) lefs than two femicircles; therefore

therefore the meafures of the angles A, B, C are greater than a femicircle; and hence the angles A, B, C are greater than two right angles.

All the external and internal angles of any triangle are equal to fix right angles; therefore all the internal angles are lefs than fix right angles.

PROP. XII. Fig. 8.

IF from any point C, which is not the pole of the great circle ABD, there be drawn arches of great circles CA, CD, CE, CF, &c. the greateft of thefe is CA, which paffes through H the pole of ABD, and CB the remainder of ACB is the leaft, and of any others CD, CE, CF, &c. CD, which is nearer to CA, is greater than CE, which is more remote.

Let the common fection of the planes of the great circles ACB, ADB be AB; and from C, draw CG perpendicular to AB, which will alfo be perpendicular to the plane ADB; (4. def. 11.) join GD, GE, GF, CD, CE, CF, CA, CB.

Of all the ftraight lines drawn from G to the circumference ADB, GA is the greateft, and GB the leaft; (7. 3.) and GD which is nearer to GA is greater than GE, which is more remote. The triangles CGA, CGD are right angled at G, and they have the common fide CG; therefore the fquares of CG, GA together, that is, the fquare of CA, is greater than the fquares of CG, GD together, that is, the fquare of CD; and CA is greater than CD, and therefore the arch CA is greater than CD. In the fame manner, fince GD is greater than GE, and GE than GF, &c. it is fhown that CD is greater than CE, and CE than CF, &c. and confequently, the arch CD greater than the arch CE, and the arch CE greater than the arch CF, &c. And fince GA is the greateft, and GB the leaft of all the ftraight lines drawn from G to the circumference ADB, it is manifeft that CA is the greateft, and CB the leaft of all the ftraight lines drawn from C to the circumference; and therefore the arch CA is the greateft, and CB the leaft of all the circles drawn through C, meeting ADB. Q. E. D.

PROP.

PROP. XIII. Fig. 9.

IN a right angled ſpherical triangle the ſides are of the ſame affection with the oppoſite angles ; that is, if the ſides be greater or leſs than quadrants, the oppoſite angles will be greater or leſs than right angles.

Let ABC be a ſpherical triangle right-angled at A, any ſide AB, will be of the ſame affection with the oppoſite angle ACB.
Caſe 1. Let AB be leſs than a quadrant, let AE be a quadrant, and let EC be a great circle paſſing through E, C. Since A is a right angle, and AE a quadrant, E is the pole of the great circle AC, and ECA a right angle ; but ECA is greater than BCA, therefore BCA is leſs than a right angle. Q. E. D.

Fig. 10.   Caſe 2. Let AB be greater than a quadrant, make AE a quadrant, and let a great circle paſs through C, E. ECA is a right angle as before, and BCA is greater than ECA, that is, greater than a right angle. Q. E. D.

PROP. XIV.

IF the two ſides of a right angled ſpherical triangle be of the ſame affection, the hypothenuſe will be leſs than a quadrant ; and if they be of different affection, the hypothenuſe will be greater than a quadrant.

Let ABC be a right angled ſpherical triangle, if the two ſides AB, AC be of the ſame or of different affection, the hypothenuſe BC will be leſs or greater than a quadrant.
Fig. 9.   Caſe 1. Let AB, AC be each leſs than a quadrant. Let AE, AG be quadrants ; G will be the pole of AB, and E the pole of AC, and EC a quadrant ; but, by Prop. 12. CE is greater than CB, ſince CB is farther off from CGD than CE. In the ſame manner, it is ſhown that CB, in the triangle CBD, where the two ſides CD, BD are each greater than a quadrant, is leſs than CE, that is, leſs than a quadrant. Q. E. D.

Caſe

Cafe 2. Let AC be lefs, and AB greater than a quadrant; Fig. 10. then the hypothenufe BC will be greater than a quadrant; for let AE be a quadrant, then E is the pole of AC, and EC will be a quadrant. But CB is greater than CE by prop. 12. fince AC paffes through the pole of ABD. Q. E. D.

PROP. XV.

IF the hypothenufe of a right-angled triangle be greater or lefs than a quadrant, the fides will be of different or the fame affection.

This is the converfe of the preceding, and demonftrated in the fame manner.

PROP. XVI.

IN any fpherical triangle ABC, if the perpendicular AD from A upon the bafe BC fall within the triangle, the angles B and C at the bafe will be of the fame affection; and if the perpendicular fall without the triangle, the angles B and C will be of different affection.

1. Let AD fall within the triangle; then (13. of this) fince Fig. 11. ADB, ADC are right-angled fpherical triangles, the angles B, C muft each be of the fame affection as AD.

2. Let AD fall without the triangle, then (13. of this) the Fig. 12. angle B is of the fame affection as AD; and by the fame, the angle ACD is of the fame affection as AD; therefore the angle ACB and AD are of different affection, and the angles B and ACB of different affection.

Cor. Hence if the angles B and C be of the fame affection, the perpendicular will fall within the bafe; for, if it did not, (16. of this), B and C would be of different affection. And if the angles B and C be of oppofite affection, the perpendicular will fall without the triangle; for, if it did not, (16. of this), the angles B and C would be of the fame affection, contrary to the fuppofition.

PROP.

PROP. XVII. FIG. 13.

IN right-angled fpherical triangles, the fine of either of the fides about the right angle, is to the radius of the fphere, as the tangent of the remaining fide is to the tangent of the angle oppofite to that fide.

Let ABC be a triangle, having the right angle at A; and let AB be either of the fides, the fine of the fide AB will be to the radius, as the tangent of the other fide AC to the tangent of the angle ABC, oppofite to AC. Let D be the centre of the fphere; join AD, BD, CD, and let AE be drawn perpendicular to BD, which therefore will be the fine of the arch AB, and from the point Æ, let there be drawn in the plane BDC the ftraight line EF at right angles to BD, meeting DC in F, and let AF be joined. Since therefore the ftraight line DE is at right angles to both EA and EF, it will alfo be at right angles to the plane AEF, (4. 11.) wherefore the plane ABD, which paffes through DE, is perpendicular to the plane AEF, (18. 11.) and the plane AEF perpendicular to ABD: The plane ACD or AFD is alfo perpendicular to the fame ABD: Therefore the common fection, viz. the ftraight line AF, is at right angles to the plane ABD: (19. 11.) And FAE, FAD are right angles; (3. def. 11.) therefore AF is the tangent of the arch AC; and in the rectilineal triangle AEF, having a right angle at A, AE will be to the radius as AF to the tangent of the angle AEF, (1. Pl. Tr.); but AE is the fine of the arch AB, and AF the tangent of the arch AC, and the angle AEF is the inclination of the planes CBD, ABD, (6. def. 11.) or the fpherical angle ABC: Therefore the fine of the arch AB is to the radius as the tangent of the arch AC, to the tangent of the oppofite angle ABC.

Cor. 1. If therefore of the two fides, and an angle oppofite to one of them, any two be given, the third will alfo be given.

Cor. 2. And fince by this propofition the fine of the fide AB is to the radius, as the tangent of the other fide AC to the

tangent

tangent of the angle ABC oppofite to that fide ; and as the radius is to the co-tangent of the angle ABC, fo is the tangent of the fame angle ABC to the radius, (Cor. 2. def. Pl. Tr.) by equality, the fine of the fide AB is to the co-tangent of the angle ABC adjacent to it, as the tangent of the other fide AC to the radius.

PROP. XVIII. Fig. 13.

IN right angled fpherical triangles the fine of the hypo-thenufe is to the radius, as the fine of either fide is to the fine of the angle oppofite to that fide.

Let the triangle ABC be right angled at A, and let AC be either of the fides ; the fine of the hypothenufe BC will be to the radius as the fine of the arch AC is to the fine of the angle ABC.

Let D be the centre of the fphere, and let CG be drawn per-pendicular to DB, which will therefore be the fine of the hy-pothenufe BC ; and from the point G let there be drawn in the plane ABD the ftraight line GH perpendicular to DB, and let CH be joined : CH will be at right angles to the plane ABD, as was fhown in the preceding propofition of the ftraight line FA : Wherefore CHD, CHG are right angles, and CH is the fine of the arch AC ; and in the triangle CHG, having the right angle CHG, CG is to the radius as CH to the fine of the angle CGH : (1. Pl. Tr.) But fince CG, HG are at right angles to DGB, which is the common fection of the planes CBD, ABD, the angle CGH will be equal to the inclination of thefe planes ; (6. def. 11.) that is, to the fpherical angle ABC. The fine, therefore, of the hypothenufe CB is to the radius as the fine of the fide AC is to the fine of the oppofite angle ABC. Q. E. D.

Cor. Of thefe three, viz. the hypothenufe, a fide, and the angle oppofite to that fide, any two being given, the third is alfo given by prop. 2.

I l 4   PROP.

PROP. XIX. Fig. 14.

IN right-angled, spherical triangles, the co-sine of the
   hypothenuse is to the radius as the co-tangent of ei-
ther of the angles is to the tangent of the remaining
angle.

Let ABC be a spherical triangle, having a right angle at A,
the co-sine of the hypothenuse BC will be to the radius as the
co-tangent of the angle ABC to the tangent of the angle ACB.
Describe the circle DE, of which B is the pole, and let it
meet AC in F, and the circle BC in E; and since the circle BD
passes through the pole B of the circle DF, DF will also pass
through the pole of BD. (13. 18. 1. Theod. sph.) And since
AC is perpendicular to BD, AC will also pass through the pole
of BD; wherefore the pole of the circle BD will be found in
the point where the circles AC, DE meet, that is, in the point
F: The arches FA, FD are therefore quadrants, and likewise
the arches BD, BE: In the triangle CEF, right-angled at the
point E, CE is the complement of the hypothenuse BC of the
triangle ABC, EF is the complement of the arch ED, which is
the measure of the angle ABC, and FC the hypothenuse of the
triangle CEF, is the complement of AC, and the arch AD,
which is the measure of the angle CFE, is the complement of
AB.
   But (17. of this) in the triangle CEF, the sine of the side
CE is to the radius, as the tangent of the other side is to the
tangent of the angle ECF opposite to it, that is, in the triangle
ABC, the co-sine of the hypothenuse BC is to the radius, as the
co-tangent of the angle ABC is to the tangent of the angle ACB.
Q. E. D.
   Cor. 1. Of these three, viz. the hypothenuse and the two
angles, any two being given, the third will also be given.
   Cor. 2. And since by this proposition the co-sine of the
hypothenuse BC is to the radius as the co-tangent of the
angle ABC to the tangent of the angle ACB. But as the ra-
dius is to the co-tangent of the angle ACB, so is the tangent
of the same to the radius; (Cor. 2. def. Pl. Tr.) and, ex
aequo, the co-sine of the hypothenuse BC is to the co-tangent

of

of the angle ACB, as the co-tangent of the angle ABC to the radius.

PROP. XX. Fig. 14.

IN right-angled spherical triangles, the co-fine of an angle is to the radius, as the tangent of the fide adjacent to that angle is to the tangent of the hypothenufe.

The fame conftruction remaining; in the triangle CEF, (17. of this) the fine of the fide EF is to the radius, as the tangent of the other fide CE is to the tangent of the angle CFE, oppofite to it; that is, in the triangle ABC, the co-fine of the angle ABC is to the radius as (the co-tangent of the hypothenufe BC to the co-tangent of the fide AB, adjacent to ABC, or as) the tangent of the fide AB to the tangent of the hypothenufe, fince the tangents of two arches are reciprocally proportional to their co-tangents. (Cor. 1. def. Pl. Tr.)

Cor. And fince by this propofition the co-fine of the angle ABC is to the radius, as the tangent of the fide AB is to the tangent of the hypothenufe BC; and as the radius is to the co-tangent of BC, fo is the tangent of BC to the radius; by equality, the co-fine of the angle ABC will be to the co-tangent of the hypothenufe BC, as the tangent of the fide AB, adjacent to the angle ABC, to the radius.

PROP. XXI. Fig. 14.

IN right-angled spherical triangles, the co-fine of either of the fides is to the radius, as the co-fine of the hypothenufe is to the co-fine of the other fide.

The fame conftruction remaining; in the triangle CEF, the fine of the hypothenufe CF is to the radius, as the fine of the fide CE to the fine of the oppofite angle CFE; (18. of this) that is, in the triangle ABC the co-fine of the fide CA is to the radius as the co-fine of the hypothenufe BC to the co-fine of the other fide BA. Q. E. D.

PROP.

P R O P. XXII. Fig. 14.

IN right angled fpherical triangles, the co-fine of either of the fides is to the radius, as the co-fine of the angle oppofite to that fide is to the fine of the other angle.

The fame conftruction remaining; in the triangle CEF, the fine of the hypothenufe CF is to the radius as the fine of the fide EF is to the fine of the angle ECF oppofite to it; that is, in the triangle ABC, the co-fine of the fide CA is to the radius, as the co-fine of the angle ABC oppofite to it, is to the fine of the other angle. Q. E. D.

Of the CIRCULAR PARTS.

IN any right angled fpherical triangle ABC, the complement Fig. 15. of the hypothenufe, the complements of the angles, and the two fides, are called *The circular parts of the triangle*, as if it were following each other in a circular order, from whatever part we begin: Thus, if we begin at the complement of the hypothenufe, and proceed towards the fide BA, the parts following in order will be the complement of the hypothenufe, the complement of the angle B, the fide BA the fide AC, (for the right angle at A is not reckoned among the parts), and, laftly, the complement of the angle C. And thus at whatever part we begin, if any three of thefe five be taken, they either will be all contiguous or adjacent, or one of them will not be contiguous to either of the other two: In the firft cafe, the part which is between the other two is called the *Middle part*, and the other two are called *Adjacent extremes*. In the fecond cafe, the part which is not contiguous to either of the other two is called the *Middle part*, and the other two *Oppofite extremes*. For example, if the three parts be the complement of the hypothenufe BC, the complement of the angle B, and the fide BA; fince thefe three are contiguous to each other, the complement of the angle B will be the middle part, and the complement of the hypothenufe BC and the fide BA will be adjacent extremes: But if the complement of the hypothenufe BC, and the fides BA, AC be taken; fince the complement of the hypothenufe is not adjacent to either of the fides, viz. on account of the complements of the two angles B and C intervening between it and the fides, the complement of the hypothenufe BC will be the middle part, and the fides, BA, AC oppofite extremes. The moft acute and ingenious Baron Napier, the inventor of Logarithms, contrived the two following rules concerning thefe parts, by means of which all the cafes of right-angled fpherical triangles are refolved with the greateft eafe.

R U L E   I.

The rectangle contained by the radius and the fine of the middle

middle part, is equal to the rectangle contained by the tangents of the adjacent parts.

<center>**R U L E II.**</center>

The rectangle contained by the radius, and the sine of the middle part is equal to the rectangle contained by the co-sines of the opposite parts.

These rules are demonstrated in the following manner.

Fig. 16. First, Let either of the sides, as BA, be the middle part, and therefore the complement of the angle B, and the side AC will be adjacent extremes. And by Cor. 2 prop. 17. of this S, BA is to the Co-T, B, as T, AC is to the radius, and therefore R×S, BA=Co-T, B×T, AC.

The same side BA being the middle part, the complement of the hypothenuse, and the complement of the angle C, are opposite extremes ; and by prop. 18. S, BC is to the radius, as S, BA to S, C ; therefore R×S, BA=S, BC×S, C.

Secondly, Let the complement of one of the angles, as B, be the middle part, and the complement of the hypothenuse, and the side BA will be adjacent extremes : And by Cor. prop. 20. Co-S, B is to Co-T, BC, as T, BA is to the radius, and therefore R×Co-S B=Co-T, BC×T, BA.

Again, Let the complement of the angle B be the middle part, and the complement of the angle C, and the side AC will be opposite extremes : And by prop. 22. Co-S, AC is to the radius, as Co S, B is to S, C : And therefore R×Co S, B=CoS, AC×S, C.

Thirdly, Let the complement of the hypothenuse be the middle part, and the complements of the angles B, C, will be adjacent extremes : But by cor. 2. prop. 19. Co S, BC is to Co-T, C as Co-T, B to the radius : Therefore R×Co S BC=Co-T, B×Co-T, C.

. Again, Let the complement of the hypothenuse be the middle part, and the sides AB, AC will be opposite extremes : But by prop. 21. Co-S, AC is to the radius, Co-S, BC to Co-S, BA ; therefore R×Co-S, BC=Co-S, BA×Co-S, AC. Q. E. D.

<center>S O L U</center>

Fig. 16.

SOLUTION of the Sixteen CASES of right-angled Spherical Triangles.

GENERAL PROPOSITION.

IN a right angled fpherical triangle, of the three fides, and three angles, any two being given befides the right angle, the other three may be found.

In the following table the folutions are derived from the preceding propofitions. It is obvious that the fame folutions may be derived from Baron Napier's two rules above demonftrated, which, as they are eafily remembered, are commonly ufed in practice.

Cafe	Given	Sought	
1	AC, C	B	R : CoS, AC :: S, C CoS, B : And B is of the fame fpecies with CA, by 22. and 13.
2	AC, B	C	CoS, AC : R :: CoS, B : S, C : By 22.
3	B, C	AC	S, C : CoS, B :: R : CoS, AC : By 22. and AC is of the fame fpecies with B. 13.
4	BA, AC	BC	R : CoS, BA :: CoS, AC : CoS, BC 21. and f both BA, AC be greater or lefs than a quadrant, BC will be lefs than a quadrant. But if they be of different affection, BC will be greater than a quadrant. 14.
5	BA, BC	AC	CoS, BA : R :: CoS, BC : CoS, AC 21. and if BC be greater or lefs than a quadrant, BA, AC will be of different or the fame affection : By 15.
6	BA, AC	B	S, BA : R :: T, CA : T, B. 17. and B is of the fame affection with AC, 13.

Cafe

Cafe	Given	Sought	
7	BA, B	AC	R : S, BA : : T, B : T, AC. 17. And AC is of the fame affection with B. 13.
8	AC, B	BA	T, B : R : : T, CA : S, BA. 17.
9	BC, C	AC	R : CoS, C : : T, BC : T, CA. 20. If BC be lefs or greater than a quadrant, C and B will be of the fame or different affection. 15. 13.
10	AC, C	BC	CoS, C : R : : T, AC : T, BC. 20. And BC is lefs or greater than a quadrant, according as C and AC or C and B are of the fame or different affection. 14. 1.
11	BC, CA	C	T, BC : R : : T, CA : CoS, C. 20. If BC be lefs or greater than a quadrant, CA and AB, and therefore CA and C, are of the fame or different affection. 15.
12	BC, B	AC	R : S, BC : : S, B : S, AC. 18. And AC is of the fame affection with B.
13	AC, B	BC	S, B : S, AC : : R : S, BC : 18.
14	BC, AC	B	S, BC : R : : S, AC : S, B : 18. And B is of the fame affection with AC.
15	B, C	BC	T, C : R : : CoT, B : CoS, BC. 19. And according as the angles B and C are of different or the fame affection, BC will be greater or lefs than a quadrant. 14.
16	BC, C	B	R : CoS, BC : : T, C : CoT, B. 19. If BC be lefs or greater than a quadrant, C and B will be of the fame or different affection. 15.

The fecond, eighth, and thirteenth cafes, which are commonly called ambiguous, admit of two folutions : For in thefe it is not determined whether the fide or meafure of the angle fought be greater or lefs than a quadrant.

PROP. XXIII. Fig. 16.

IN fpherical triangles, whether right angled or oblique angled, the fines of the fides are proportional to the fines of the angles oppofite to them.

First, Let ABC be a right-angled triangle, having a right angle at A ; therefore by prop. 18. the fine of the hypothenufe BC is to the radius (or the fine of the right angle at A) as the fine of the fide AC to the fine of the angle B. And, in like manner, the fine of BC is to the fine of the angle A, as the fine of AB to the fine of the angle C ; wherefore (11. 5.) the fine of the fide AC is to the fine of the angle B, as the fine of AB to the fine of the angle C.

Secondly, Let BCD be an oblique-angled triangle, the fine of either of the fides BC, will be to the fine of either of the other two CD, as the fine of the angle D oppofite to BC is to the fine of the angle B oppofite to the fide CD.   Through the point C, let there be drawn an arch of a great circle CA perpendicular upon BD ; and in the right angled triangle ABC (18. of this) the fine of BC is to the radius, as the fine of AC to the fine of the angle B ; and in the triangle ADC (by 18. of this :) And, by inverfion, the radius is to the fine of DC as the fine of the angle D to the fine of AC : Therefore, ex aequo perturbate, the fine of BC is to the fine of DC, as the fine of the angle D to the fine of the angle B.   Q. E. D.

PROP. XXIV. Fig. 17. 18.

IN oblique-angled fpherical triangles, having drawn a perpendicular arch from any of the angles upon the oppofite fide, the co-fines of the angles at the bafe are proportional to the fines of the verticle angles.

Let

. Let BCD be a triangle, and the arch CA perpendicular to the base BD; the co-fine of the angle B will be to the co-fine of the angle D, as the fine of the angle BCA to the fine of the angle DCA.

For by 22. the co-fine of the angle B is to the fine of the angle BCA as (the co-fine of the fide AC is to the radius; that is, by Prop. 22. as) the co-fine of the angle D to the fine of the angle DCA; and, by permutation, the co-fine of the angle B is to the co-fine of the angle D, as the fine of the angle BCA to the fine of the angle DCA.   Q. E. D.

P R O P. XXV. Fig. 17. 18.

THE fame things remaining, the co-fines of the fides BC, CD, are proportional to the co-fines of the bafes BA, AD.

For by 21. the co-fine of BC is to the co-fine of BA, as (the co-fine of AC to the radius; that is, by 21. as) the co-fine of CD is to the co-fine of AD: Wherefore, by permutation, the co-fines of the fides BC, CD are proportional to the co fines of the bafes BA, AD.   Q. E. D.

P R O P. XXVI. Fig. 17. 18.

THE fame conftruction remaining, the fines of the bafes BA, AD are reciprocally proportional to the tangents of the angles B and D at the bafe.

For by 17. the fine of BA is to the radius, as the tangent of AC to the tangent of the angle B; and by 17. and inverfion, the radius is to the fine of AD, as the tangent of D to the tangent of AC: Therefore, ex aequo perturbate, the fine of BA is to the fine of AD, as the tangent of D to the tangent of B.

PROP.

P R O P. XXVII. Fig. 17. 18.

THE co-fines of the vertical angles are reciprocally proportional to the tangents of the fides.

For by prop. 20. the co-fine of the angle BCA, is to the ra-dius as the tangent of CA is to the tangent of BC; and by the fame prop. 20. and by inverfion, the radius is to the co-fine of the angle DCA, as the tangent of DC to the tangent of CA : Therefore, ex aequo perturbate, the co fine of the angle BCA is to the co-fine of the angle DCA, as the tangent of DC is to the tangent of BC.   Q. E. D.

L E M M A.   Fig. 19. 20.

IN right-angled plain triangles, the hypothenufe is to the radius, as the excefs of the hypothenufe above either of the fides to the verfed fine of the acute angle adjacent to that fide, or as the fum of the hypothenufe, and either of the fides to the verfed fine of the exterior angle of the triangle.

Let the triangle ABC have a right angle at B; AC will be to the radius as the excefs of AC above AB, to the verfed fine of the angle A adjacent to AB; or as the fum of AC, AB to the verfed fine of the exterior angle CAK.

With any radius DE, let a circle be defcribed, and from D the centre let DF be drawn to the circumference, making the angle EDF equal to the angle BAC, and from the point F, let FG be drawn perpendicular to DE : Let AH, AK be made equal to AC, and DL to DE : DG therefore is the co-fine of the angle EDF or BAC, and GE its verfed fine : And becaufe of the equiangular triangles ACB, DFG, AC or AH is to DF or DE, as AB to DG : Therefore (19. 5.) AC is to the radius DE as BH to GE, the verfed fine of the angle EDF or BAC : And fince AH is to DE, as AB to DG, (12. 5.) AH or AC will be to the radius DE as KB to LG, the verfed fine of the angle LDF or KAC.   Q. E. D.

PROP. XXVIII. Fig. 21. 22.

IN any spherical triangle, the rectangle contained by the fines of two fides, is to the fquare of the radius, as the excefs of the verfed fines of the third fide or bafe, and the arch, which is the excefs of the fides, is to the verfed fine of the angle oppofite to the bafe.

Let ABC be a fpherical triangle, the rectangle contained by the fines of AB, BC will be to the fquare of the radius, as the excefs of the verfed fines of the bafe AC, and of the arch, which is the excefs of AB, BC to the verfed fine of the angle ABC oppofite to the bafe.

Let D be the centre of the fphere, and let AD, BD, CD be joined, and let the fines AE, CF, CG of the arches AB, BC, AC be drawn; let the fide BC be greater than BA, and let BH be made equal to BC; AH will therefore be the excefs of the fides BC, BA; let HK be drawn perpendicular to AD, and fince AG is the verfed fine of the bafe AC, and AK the verfed fine of the arch AH, KG is the excefs of the verfed fines of the bafe AC, and of the arch AH, which is the excefs of the fides BC, BA: Let GL likewife be drawn parallel to KH, and let it meet FH in L, let CL, DH be joined, and let AD, FH meet each other in M.

Since therefore in the triangles CDF, HDF, DC, DH are equal, DF is common, and the angle FDC equal to the angle FDH, becaufe of the equal arches BC, BH, the bafe HF will be equal to the bafe FC, and the angle HFD equal to the right angle CFD: The ftraight line DF therefore (4. 11.) is at right angles to the plane CFH: Wherefore the plane CFH is at right angles to the plane BDH, which paffes through DF. (18. 11.) In like manner, fince DG is at right angles to both GC and GL, DG will be perpendicular to the plane CGL; therefore the plane CGL is at right angles to the plane BDH, which paffes through DG: And it was fhown, that the plane CFH or CFL, was perpendicular to the fame plane BDH; therefore the common fection of the planes CFL, CGL, viz. the ftraight line GL, is perpendicular to the plane BDA, (19. 11.) and therefore CLF is a right angle: In the triangle CFL having the right angles CLF, by the lemma CF
is

is to the radius as LH, the excefs, viz. of CF or FH above
FL, is to the verfed fine of the angle CFL; but the angle
CFL is the inclination of the planes BCD, BAD, fince FC,
FL are drawn in them at right angles to the common fection
BF: The fpherical angle ABC is therefore the fame with the
angle CFL; and therefore CF is to the radius as LH to the
verfed fine of the fpherical angle ABC; and fince the triangle
AED is equiangular (to the triangle MFD, and therefore) to
the triangle MGL, AE will be to the radius of the fphere AD,
(as MG to ML; that is, becaufe of the parallels as) GK to LH:
The ratio therefore which is compounded of the ratios of AE
to the radius, and of CF to the fame radius; that is, (23. 6.)
the ratio of the rectangle contained by AE, CF to the fquare of
the radius, is the fame with the ratio compounded of the ratio
of GK to LH, and the ratio of LH to the verfed fine of the
angle ABC; that is, the fame with the ratio of GK to the ver-
fed fine of the angle ABC; therefore, the rectangle contained
by AE, CF, the fines of the fides AB, BC, is to the fquare of
the radius as GK, the excefs of the verfed fines AG, AK, of
the bafe AC, and the arch AH, which is the excefs of the fides
to the verfed fine of the angle ABC oppofite to the bafe AC.
Q. E. D.

P R O P. XXIX. Fig. 23.

THE rectangle contained by half of the radius, and
the excefs of the verfed fines of two arches, is
equal to the rectangle contained by the fines of half the
fum, and half the difference of the fame arches.

Let AB, AC be any two arches, and let AD be made equal
to AC the lefs; the arch DB therefore is the fum, and the arch
CB the difference of AC, AB: Through E the centre of the
circle, let there be drawn a diameter DEF, and AE joined, and
CD likewife perpendicular to it in G; and let BH be perpendi-
cular to AE, and AH will be the verfed fine of the arch AB,
and AG the verfed fine of AC, and HG the excefs of thefe ver-
fed fines: Let BD, BC, BF be joined, and FC alfo meeting
BH in K.
Since therefore BH, CG are parallel, the alternate angles
BKC, KCG will be equal; but KCG is in a femicircle, and
<div align="center">K k 2</div> therefore

therefore a right angle ; therefore BKC is a right angle ; and in the triangles DFB, CBK, the angles FDB, BCK in the same segment are equal, and FBD, BKC are right angles ; the triangles DFB, CBK are therefore equiangular ; wherefore DF is to DB, as BC to CK, or HG ; and therefore the rectangle contained by the diameter DF, and HG is equal to that contained by DB, BC ; wherefore the rectangle contained by a fourth part of the diameter, and HG, is equal to that contained by the halves of DB, BC : But half the chord DB is the fine of half the arch DAB, that is, half the fum of the arches AB, AC ; and half the chord of BC is the fine of half the arch BC, which is the difference of AB, AC. Whence the propofition is manifeft.

PROP. XXX. Fig. 19. 24.

THE rectangle contained by half of the radius, and the verfed fine of any arch, is equal to the fquare of the fine of half the fame arch.

Let AB be an arch of a circle, C its centre, and AC, CB, BA being joined : Let AB be bifected in D, and let CD be joined, which will be perpendicular to BA, and bifect it in E. (4. 1.) BE or AE therefore is the fine of the arch DB or AD, the half of AB : Let BF be perpendicular to AC, and AF will be the verfed fine of the arch BA ; but, becaufe of the fimilar triangles CAE, BAF, CA is to AE as AB, that is, twice AE to AF ; and by halving the antecedents, half of the radius CA is to AE the fine of the arch AD, as the fame AE to AF the verfed fine of the arch AB. Wherefore by 16. 6. the propofition is manifeft.

PROP. XXXI. Fig. 25.

IN a fpherical triangle, the rectangle contained by the fines of the two fides, is to the fquare of the radius, as the rectangle contained by the fine of the arch which is half the fum of the bafe, and the excefs of the fides, and the fine of the arch, which is half the difference of the fame to the fquare of the fine of half the angle oppofite to the bafe.

Let

. Let ABC be a fpherical triangle, of which the two fides are. AB, BC, and bafe AC, and let the lefs fide BA be produced, fo that BD fhall be equal to BC: AD therefore is the excefs of BC, BA; and it is to be fhown, that the rectangle contained by the fines of BC, BA is to the fquare of the radius, as the rectangle contained by the fine of half the fum of AC, AD, and the fine of half the difference of the fame AC, AD to the fquare of the fine of half the angle ABC, oppofite to the bafe AC.

Since by prop. 28. the rectangle contained by the fines of the fides BC, BA is to the fquare of the radius, as the excefs of the verfed fines of the bafe AC and AD, to the verfed fine of the angle B; that is, (1. 6.) as the rectangle contained by half the radius, and that excefs, to the rectangle contained by half the radius, and the verfed fine of B; therefore (29. 30. of this), the rectangle contained by the fines of the fides BC, BA is to the fquare of the radius, as the rectangle contained by the fine of the arch, which is half the fum of AC, AD, and the fine of the arch which is half the difference of the fame AC, AD is to the fquare of the fine of half the angle ABC.   Q. E. D.

SOLUTION

SOLUTION of the twelve CASES of oblique-angled SPHERICAL TRIANGLES.

GENERAL PROPOSITION.

IN an oblique angled spherical triangle, of the three sides and three angles, any three being given, the other three may be found.

Fig. 26. 27.

	Given	Sought	
1	B, D, and BC, two angles and a side opposite one of them.	C.	CoS, BC : R :: CoT, B : T, BCA. 19. Likewise by 24. CoS, B : S, BCA :: CoS, D : S, DCA ; wherefore BCD is the sum or difference of the angles DCA, BCA according as the perpendicular CA falls within or without the triangle BCD ; that is, (16. of this,) according as the angles B, D are of the same or different affection.
2	B, C, and BC two angles and the side between them.	D.	CoS, BC : R :: CoT, B : T, BCA, 19. and also by 24. S, BCA : S, DCA :: CoS, B : CoS, D ; and according as the angle BCA is lefs or greater than BCD. the perpendicular CA falls within or without the triangle BCD ; and therefore (16. of this,) the angles B, D will be of the same or different affection.
3	BC, CD, and B.	BD.	R : CoS, B :: T, BC : T, BA. 20. and CoS, BC. : CoS, BA : : CoS, DC : CoS, DA. 25. and BD is the fum or difference of BA, DA.
4	BC, DB and B.	CD.	R : CoS, B : : T, BC : T, BA. 20. and CoS, BA : CoS, BC : : CoS, DA : CoS, DC. 25. and according as DA, AC are of the fame or different affection, DC will be lefs or greater than a quadrant. 14.

	Given.	Sought.	
5	B, D and BC.	DB.	R : Co S, B :: T, BC : T, BA. 20. and T, D : T, B :: S, BA : S, DA 26. and BD is the fum or difference of BA DA.
6	BC, BD and B.	D.	R : Co S, B :: T, BC : T, BA. 20. and S, DA : S, BA :: T, B : T, D ; and according as BD is greater or lefs than BA, the angles B, D are of the fame or different affection. 16.
7	BC, DC and B.	C.	Co S, BC : R :: Co T, B : T, BCA. 19. and T, DC : T, BC :: Co S, BCA : Co S, DCA. 27. the fum or difference of the angles BCA, DCA is equal to the angle BCD.
8	B, C and BC.	DC.	Co S, BC : R :: Co T, B : T, BCA 19. alfo by 27. Co S, DCA : Co S, BCA :: T, BC : T, DC. 27. if DCA and B be of the fame affection; that is, (13.) if AD and CA be fimilar, DC will be lefs than a quadrant, 14. and if AD, CA be not of the fame affection, DC is greater than a quadrant. 14.
9	BC, DC and B.	D.	S, CD : S, B :: S, BC : S, D.
10	B, D and BC.	DC.	S, D : S, BC :: S, B : S, DC.
11	BC, BA, AC. Fig. 25.	B.	S, AB × S, BC : Rq :: S, $\frac{AC + AD}{2}$ × S, $\frac{AC-AD}{2}$: Sq, $\frac{ABC}{2}$. See Fig. 25 AD being the difference of the fides BC, BA.

Given.   Sought.

	Given.	Sought.	
12	A, B, C. Fig. 7.	The fides.	See Fig. 7. In the triangle DEF, DE, EF, FD are refpectively the fupplements of the meafures of the given angles B, A, C in the triangle BAC; the fides of the triangle DEF are therefore given, and by the preceding cafe the angles D, E, F may be found, and the fides BC, BA, AC are the fupplements of the meafures of thefe angles.

The 3d, 5th, 7th, 9th, 10th, cafes, which are commonly called ambiguous, admit of two folutions, either of which will anfwer the conditions required; for, in thefe cafes, the meafure of the angle or fide fought, may be either greater or lefs than a quadrant, and the two folutions will be fupplements to each other. (cor. to def. 4. 6. Pl. Tr.)

If from any of the angles of an oblique-angled fpherical triangle, a perpendicular arch be drawn upon the oppofite fide, moft of the cafes of oblique angled triangles may be refolved by means of Napier's rules.

F I N I S.